The net measure properties of symmetric Cantor sets and their applications *

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Abstract The equivalence between Hausdorff measure induced by natural covering net and Hausdorff measure in usual meaning has been obtained for one-dimensional symmetric Cantor sets. As an application, the Hausdorff dimensions of such sets are determined.

Keywords: symmetric Cantor sets, net measure, Hausdorff measure, Hausdorff dimension.

Let $\mathscr{F}$ be a collection of subsets of interval $\boldsymbol{E}_0=[0,1]$ such that for any $x\in\boldsymbol{E}_0$ and $\varepsilon>0$, there exists $I\in\mathscr{F}$ with $x\in I$ and $|I|\leqslant\varepsilon$, where $|I|$ denotes the diameter of I. $\mathscr{F}$ is called a net for $\boldsymbol{E}_0$.

Given $E\subset[0,1]$ and α, $\varepsilon>0$, define

$$H^{\alpha}_{\varepsilon,\mathscr{F}}(E)=\inf\left\{\sum_j|I_j|^{\alpha};E\subset\bigcup_j I_j,I_j\in\mathscr{F},|I_j|\leqslant\varepsilon\right\},$$

$$H^{\alpha}_{\mathscr{F}}(E)=\lim_{\varepsilon\to 0}H^{\alpha}_{\varepsilon,\mathscr{F}}(E),$$

$$\dim_{H,\mathscr{F}}(E)=\inf\{\alpha>0;\ H^{\alpha}_{\mathscr{F}}(E)=0\}.$$

$H^{\alpha}_{\mathscr{F}}(E)$ and $\dim_{H,\mathscr{F}}(E)$ are called α-dimensional Hausdorff measure and Hausdorff dimension of E with respect to $\mathscr{F}$ respectively. If $\mathscr{F}$ consists of all subsets of $\boldsymbol{E}_0$, then we just get the Hausdorff measure and Hausdorff dimension in usual sense.

Now let $\mathscr{F}_1$, $\mathscr{F}_2$ be two nets for $\boldsymbol{E}_0$, and $E\subset\boldsymbol{E}$ Then $\mathscr{F}_1$, $\mathscr{F}_2$ are called equivalent for E if there exist two positive constants, c_1, c_2, such that for any $0<\alpha\leqslant 1$,

$$c_1H^{\alpha}_{\mathscr{F}_1}(E)\leqslant H^{\alpha}_{\mathscr{F}_2}(E)\leqslant c_2H^{\alpha}_{\mathscr{F}_1}(E).$$

In this case, it is easy to see that the Hausdorff dimensions induced respectively by these two nets for E are equal.

We have pointed out that the net related to Hausdorff measure in usual sense just consists of all subsets of $\boldsymbol{E}_0$. Thus if we want to calculate the Hausdorff dimension of E by the definition, we must consider all possible covers for set E. But if we can find a well-constructed set equivalent

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to the above net, then we might simplify the calculation for measure and dimension. For more discussions about net measure, please refer to references [1—4].

In this paper we discuss the net measure properties of symmetric Cantor sets, the Hausdorff dimension of which, as an application, can be determined.

Now we begin to define symmetric Cantor sets as follows.

Let $\mathcal{N}=\{n_k\}_{k\geqslant 1}$ be a sequence of positive integers and let $\mathcal{C}=\{c_k\}$ be a sequence of positive real numbers satisfying $n_k\geqslant 2$, $0<c_k<1$ and $n_k c_k\leqslant 1$ for all k.

Now we construct E_1 from $\boldsymbol{E}_0$. E_1 is a union of n_1 closed subintervals of $\boldsymbol{E}_0$ satisfying: (i) these n_1 intervals have the same length c_1; (ii) the gaps between these intervals are the same; (iii) the leftmost interval has the same left end-point with $\boldsymbol{E}_0$. The rightmost one has the same right end-point with $\boldsymbol{E}_0$. These n_1 intervals are called fundamental intervals of order 1.

Let $\mathcal{F}_1$ denote the collection of all fundamental intervals of order 1. We will define E_2, E_3, $\cdots$ by induction. Now assume that E_k is defined, and let $\mathcal{F}_k$ be the collection of all fundamental intervals of order k. Letting $I\in\mathcal{F}_k$ be any fundamental interval of order k, we replace $\boldsymbol{E}_0$ by I, and repeat the process of constracting E_1 from $\boldsymbol{E}_0$. We get E_{k+1} when I runs over $\mathcal{F}$ (noticing that the length of each fundamental interval of order $k+1$ is $c_{k+1}|I|=\prod_{j=1}^{k+1}c_j$).

Finally, let $E=\bigcap_{k\geqslant 0}E_k$ and call it a symmetric Cantor set determined by the sequences $\mathcal{N}$, $\mathcal{C}$ Denote $\overline{E}=E(\mathcal{N},\ \mathcal{C})$. Symmetric Cantor sets are very important Fractal sets, some special types of which had been studied by Kahane *et al*.[5], Lee *et al*.[6], Moorthy *et al*.[7] and Hua Su[2]. The Hausdorff dimensions of some special cases have been determined. But the Hausdorff dimensions of general cases have not been obtained up to now. In this paper we will solve this problem by the net measure techniques.

1 The net measure properties of symmetric Cantor sets

Let $\mathcal{F}_0$ denote the collection of all subsets of $\boldsymbol{E}_0$. As we have discussed above, the net measure technique consists in finding a relatively simple net which is equivalent to $\mathcal{F}_0$. From the definition of symmetric Cantor sets, we know that $\{\mathcal{F}_k\}_{k\geqslant 1}$ is a simple and natural net. We will show that it is equivalent to $\mathcal{F}_0$ by using some intermediate nets. Thus it will be easy to determine the dimensions of symmetric Cantor sets.

Now we give some definitions and notations as follows.

Let $\mathcal{V}=\{v_i\}$ be a set sequence (finite or countable infinite). Let $A\subset\boldsymbol{E}_0$ and define

$$A\cap\mathcal{V}=\{A\cap v_j: v_j\in\mathcal{V}\},\ \|\mathcal{V}\|^s=\sum_{v_j\in\mathcal{V}}|v_j|^s,\ \mathcal{V}+x=\{v_j+x\}_{v_j\in\mathcal{V}},$$

where $v_j+x=\{y+x: y\in v_j\}$. Let $\mathcal{G}_k=\{I: I=\bigcup_{i=1}^{m}I_i,\ I_i\in\mathcal{F}_k,\ 1\leqslant m\leqslant n_k$, and $\exists\, J\in\mathcal{F}_{k-1}$ such that $I\subset J\}$, i.e. any element of G_k is a union of some fundamental intervals of order k which are generated from the same fundamental interval of order $k-1$. Let $\mathscr{S}=\bigcup_{k\geqslant 1}\mathscr{S}_k$.

Lemma 1. *The nets $\mathscr{F}_0$ and $\mathscr{S}$ are equivalent.*

Proof. $\forall\, 0<s\leqslant 1$, $H^s_{\mathscr{F}_0}(E)\leqslant H^s_{\mathscr{S}}(E)$ since $\mathscr{S}$ is a sub-collection of $\mathscr{F}_0$. So we only need to show that there exists $c>0$ such that

$$cH^s_{\mathscr{S}}(E)\leqslant H^s_{\mathscr{F}_0}(E). \tag{1}$$

Let I be an open sub-interval of $\boldsymbol{E}_0$, $I\cap E\neq\emptyset$. Then there is only one positive integer k such that I contains at least one fundamental interval of order k, but no fundamental interval of order $k-1$. Hence I intersects at most two fundamental intervals of order $k-1$.

If I intersects two fundamental intervals of order $k-1$, we denote them by J_1, J_2. Let $I_1=J_1\cap I$, $I_2=J_2\cap I$. Since $|I_1|^s\leqslant|I|^s$, $|I_2|^s\leqslant|I|^s$, we have

$$|I|^s\geqslant\frac{1}{2}(|I_1|^s+|I_2|^s). \tag{2}$$

Let $G(I_i)$ be the union of fundamental intervals of order k which intersect I_i, $i=1,2$. Then $G(I_i)\in\mathscr{S}_k$, and $|G(I_i)|\leqslant|I_i|+2\prod_{i=1}^{k}c_i$. Since one of I_1 and I_2, or both I_1 and I_2, contain a fundamental interval of order k, we can assume that I_1 is the interval. So

$$|G(I_1)|^s\leqslant(|I_1|+2\prod_{i=1}^{k}c_i)^s\leqslant(3|I_1|)^s. \tag{3}$$

If I_2 contains also one fundamental interval of order k, then by the same argument, we have

$$|G(I_2)|^s\leqslant(3|I_2|)^s.$$

So

$$|I|^s\geqslant\frac{3^{-s}}{2}(|G(I_1)|^s+|G(I_2)|^s).$$

If I_2 contains no fundamental interval of order k, then

$$|G(I_2)|=\prod_{i=1}^{k}c_i\leqslant|G(I_1)|.$$

So by eqs. (2) and (3), we have

$$|I|^s\geqslant\frac{3^{-s}}{2}|G(I_1)|^s\geqslant\frac{3^{-s}}{4}(|G(I_1)|^s+|G(I_2)|^s).$$

So we always have

$$|I|^s\geqslant\frac{3^{-s}}{4}(|G(I_1)|^s+|G(I_2)|^s)\geqslant\frac{1}{12}(|G(I_1)|^s+|G(I_2)|^s). \tag{4}$$

When I intersects only one fundamental interval of order $k-1$, by similar discussion, we have

$$|I|^s \geqslant 3^{-s}|G(I)|^s, G(I) \in G_k. \tag{5}$$

By the above analysis, we can see that $I \cap E \subset (G(I_1) \cap E) \cup (G(I_2) \cap E)$, or $(I \cap E \subset G(I) \cap E)$, and $|G(I_1)|$, $|G(I_2)| \leqslant 6|I|$.

Now let $I = \{I_j\}$ be a δ-cover of E. By the above discussions and eqs. (4) and (5), we may find a 6δ-cover $\mathscr{S}^* = \{g_i\} \in \mathscr{S}$ such that

$$\| I \|^s \geqslant \frac{1}{12} \| \mathscr{S}^* \|^s.$$

So letting $c = \frac{1}{12}$, we obtain (1). Q.E.E.

Now we consider a fundamental interval $I \in \mathscr{F}_{k-1}$, and denote the n_k fundamental intervals of order k generated from I by $I_{k,1}(I), \cdots, I_{k,n_k}(I)$ in order. Let m be an integer, $1 \leqslant m < n_k$, and let

$$n_k = qm + r, \quad 0 \leqslant r < m, q \in \mathbb{N}.$$

If $r > 0$, we construct $q+1$ subsets of I as follows.

Let $\widetilde{W}_{k,m,1}(I)$ be the closed interval which has the same left endpoint with $I_{k,1}(I)$ and the same right endpoint with $I_{k,m}(I)$. $W_{k,m,1} = \bigcup_{A \in \widetilde{W}_{k,m,1} \cap \mathscr{F}_k} A$.

Let $\widetilde{W}_{k,m,q}(I)$ be the closed interval which has the same left endpoint with $I_{k,(q-1)m+1}(I)$ and the same right endpoint with $I_{k,q \cdot m}(I)$. $W_{k,m,q} = \bigcup_{A \in \widetilde{W}_{k,m,q} \cap \mathscr{F}_k} A$.

Let $\widetilde{W}_{k,m,q+1}(I)$ be the closed interval which has the same left endpoint with $I_{k,(qm+1)}(I)$ and the same right endpoint with $I_{k,qm+r}(I)$. $W_{k,m,q+1} = \bigcup_{A \in \widetilde{W}_{k,m,q+1} \cap \mathscr{F}_k} A$.

Put $\mathscr{W}_{k,m}(I) = \{W_{k,m,j}\}_{1<j<q+1}$, $\mathscr{W}_{k,m} = \{v: v \in \mathscr{W}_{k,m}(I), I \in \mathscr{F}_{k-1}\}$.

We see that $\mathscr{W}_{k,m}$ is a cover collection of E_k, and for any k, m, $\mathscr{W}_{k,m} \subset \mathscr{S}$.

If $r = 0$, in the same way, we construct $q+1$ subsets of I except that the last one is empty.

Lemma 2. *There exists $c > 0$ such that for any cover collection $\mathscr{V}$ ($\subset \mathscr{S}$) of E, there exists a cover collection $\mathscr{W}_{k,m}$ such that $\forall\, 0 < s \leqslant 1$,*

$$\| \mathscr{V} \|^s \geqslant c \, \| \mathscr{W}_{k,m} \|^s.$$

Proof. Let $\mathscr{V} = \{v_j\}$ be an arbitrary cover subcollection of $\mathscr{S}$. Then we can assume that $\mathscr{V}$ is finite since E is compact. Assume that k_1 and k_2 are respectively the lowermost and highest order or fundamental intervals contained in the same v_j.

Put

$D_1 = \{v_j : v_j \in \mathscr{V}\ v_j$ contains at least one fundamental interval of order $k_1\}$;

$D_2 = \{I : I \in \mathscr{F}_{k_1}$, there exists no $v_j \in \mathscr{V}$ such that $I \subset v_j\}$;

$D = D_1 \cup D_2$.

The above definitions imply that D is a cover collection of E_{k_1}. For $p \in D$, let

$$h(p, s) = \begin{cases} \dfrac{|p|^s}{\# p(k_1)}, & \text{if } p \in D_1, \\ \| p \cap \mathscr{V} \|^s, & \text{if } p \in D_2, \end{cases}$$

where $\# p(k_1)$ denotes the number of fundamental intervals of order k_1 contained in p. By the construction of $\mathscr{S}$, we know that if $v_i \in \mathscr{V}$, and $v_j \notin D_1$, then v_j must be contained in some fundamental interval of order k_1. Assume that $h(p, s)$ attains to its minimum at $p = p_0$ (this is always possible since $\mathscr{V}$ is finite).

(i) $p_0 \in D_1$.

In this case, by the above analysis we have

$$\begin{aligned} \|\mathscr{V}\|^s &= \sum_{p \in D_1} |p|^s + \sum_{p \in D_2} \| p \cap \mathscr{V} \|^s \\ &= \sum_{p \in D_1} \# p(k_1) h(p, s) + \sum_{p \in D_2} h(p, s) \\ &\geqslant \sum_{p \in D_1} \# p(k_1) h(p_0, s) + \sum_{p \in D_2} h(p_0, s) \\ &= \Big(\sum_{p \in D_1} \# p(k_1) + \# D_2 \Big) h(p_0, s). \end{aligned}$$

Notice that $(\sum_{p \in D_1} \# p(k_1) + \# D)$ is just the number of fundamental intervals of order k_1, thus it equals $n_1 n_2 \cdots n_k$. Since

$$n_{k_1} h(p_0, s) = \frac{n_{k_1}}{\# p_0(k_1)} |P_0|^s,$$

by direct calculation, we have

$$\|\mathscr{V}\|^s \geqslant \frac{1}{2} \| \mathscr{W}_{k_1, \# p_0(k_1)} \|^s. \tag{6}$$

(ii) $p_0 \in D_2$.

In this case p_0 is a fundamental interval of order k_1. Suppose that the distances between the left endpoints of $n_1 n_2 \cdots n_k$ fundamental intervals of order k_1 and the left endpoint of p_0 are respectively $t_1, t_2, \cdots, t_{n_1 n_2 \cdots n_{k1}}$. Denote $\mathscr{V}^* = \{p_0 \cap \mathscr{V} + t_i\}_{1 \leqslant i \leqslant n_1 n_2 \cdots n_{k-1}}$. Then $\mathscr{V}^* \subset \mathscr{S}$. Since $p_0 \cap \mathscr{V}$ contains no fundamental interval of order $k-1$, any element of $\mathscr{V}$ contains no fundamental interval of order k_1. Thus for $\forall_{v_j^*} \in \mathscr{V}^*$, $\| v_j^* \|^s = h(p_0, s)$. So by the calculation similar to that of (i), we have

$$\| \mathscr{V} \|^s \geqslant \left(\prod_{j=1}^{k_1} n_j \right) h(p_0, s) = \| \mathscr{V}^* \|^s. \tag{7}$$

We have pointed out that $\mathscr{V}^* \subset \mathscr{S}$, and any element of $\mathscr{V}^*$ contains no fundamental interval of order k_1. Noticing that k_2 is the highest order of fundamental intervals contained in some elements of $\mathscr{V}^*$, we may repeat the discussion of (i), and by formulae (6)and (7), after a finite number of steps, we may find k^*, $k_1 \leqslant k^* \leqslant k_2$, and positive integer m such that

$$\| \mathscr{V} \|^s \geqslant \frac{1}{2} \| \mathscr{W}_{k^*, m} \|^s.$$

Lemma 3. *There exists a positive constant $c > 0$, such that for any positive integer k, $m \geqslant 2$ and $0 < s \leqslant 1$, there is*

$$\| \mathscr{W}_{k,m} \|^s \geqslant c \, \| \mathscr{F}_{k-1} \|^s.$$

Proof. Suppose that I is an arbitrary fundamental interval of order k-1, and let $n_k = qm + r$, $0 \leqslant r < m$. Then $n_k \leqslant (q+1)m$. Let $\delta(m)$ be the diameter of the union of m adjacent fundamental interval of order k. Then $\delta(m)$ is bigger than the gap between any two adjacent fundamental intervals of order k. Thus we have

$$|I| \leqslant 2(q+1)\delta(m).$$

So

$$|I|^s \leqslant 2^s (q+1)^s (\delta(m))^s \leqslant 4q(\delta(m))^s.$$

Hence

$$\| \mathscr{F}_{k-1} \|^s \leqslant 4 \, \| \mathscr{W}_{k,m} \|^s.$$

Remark 1. If $m = 1$, then $\mathscr{W}_{k,m} = \mathscr{W}_{k,1} = \mathscr{F}_k$, and $|\mathscr{W}_{k,m}|^s = |\mathscr{F}_k|^s$.

Theorem 1. *Let E be the symmetric Cantor set determined by the sequences $\{n_k\}_{k \geqslant 1}$, $\{c_k\}_{k \geqslant 1}$, Then*

$$c \varliminf_{k \to \infty} \prod_{i=1}^{k} n_i c_i^s \leqslant H^s(E) \leqslant \varliminf_{k \to \infty} \prod_{i=1}^{k} n_i c_i^s,$$

where $0 \leqslant s \leqslant 1$, and c is an absolute positive constant.

Proof. Note that for any $k \geqslant 1$, and $\mathscr{F}_k$ is a cover of E. So

$$H^s(E) \leqslant \varliminf_{k\to\infty} \| \mathscr{F}_k \|^s = \varliminf_{k\to\infty} \prod_{i=1}^{k} n_i c_i^s.$$

Now suppose that $\mathscr{V}$ is an arbitrary δ-cover of E. Then according to Lemmas 1—3 and Remark 1, there exist positive constant c and positive integer k such that

$$\| \mathscr{V} \|^s \geqslant c \| \mathscr{F}_k \|^s.$$

So if δ is small enough, we have $|\mathscr{V}|^s \geqslant c \varliminf_{k\to\infty} |\mathscr{F}_k|^s$, i.e. $H^s(E) \geqslant c \varliminf_{k\to\infty} \prod_{i=1}^{k} n_i c_i^s$.

Theorem 2. *Let E be the symmetric Cantor set determined by the sequences* $\{n_k\}_{k\geqslant 1}$, $\{c_k\}_{k\geqslant 1}$. *Then*

$$\dim_H E = \varliminf_{k\to\infty} \frac{\log n_1 n_2 \cdots n_k}{-\log c_1 c_2 \cdots c_k}.$$

Proof. Given any $s > \varliminf_{k\to\infty} \frac{\log n_1 n_2 \cdots n_k}{-\log c_1 c_2 \cdots c_k}$, there exists a subsequence $\{k_i\}_{i\geqslant 1}$, such that $s > \frac{\log n_1 n_2 \cdots n_{k_i}}{-\log c_1 c_2 \cdots c_{k_i}}$. So $\prod_{j=1}^{k_i} n_j c_j^s \leqslant 1$. Thus $\varliminf_{k\to\infty} \prod_{i=1}^{k} n_i c_i^s \leqslant 1$. Hence by Theorem 1, $H^s(E) \leqslant 1$. So $\dim_H(E) \leqslant s$. Thus

$$\dim_H(E) \leqslant \varliminf_{k\to\infty} \frac{\log n_1 n_2 \cdots n_k}{-\log c_1 c_2 \cdots c_k}.$$

By similar discussion, we may prove the converse inequality.

Given $E \subset \mathbb{R}$, E is called s-set if $0 < H^s(E) < \infty$. According to Theorems 1 and 2, we have

Corollary 1. *The symmetric Cantor set E is s-set if and only if* $0 < \varliminf_{k\to\infty} \prod_{i=1}^{k} n_i c_i^s < \infty$, *and* $s = \varliminf_{k\to\infty} \frac{\log n_1 n_2 \cdots n_k}{-\log c_1 c_2 \cdots c_k}$.

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