The geometry of Fractal percolation and randomly perturbed self-affine sets

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To model turbulence, Mandelbrot [6, 7] introduced a statistically self-similar family of random Cantor sets in $\mathbb{R}^d$. Since that time this family has got at least three names in the literature: `fractal percolation`, Mandelbrot percolation and canonical curdling, among which we will use the first one.

In the case of the fractal percolation random sets, on the n-th level ($1 \leq n < \infty$) of the construction, we consider some fixed (exponentially small) cubes whose positions are deterministic and the randomness comes from the fact that with some probabilities we retain or discard these cubes. In our second example of random Cantor sets we apply a construction which is similar to the construction of deterministic self-affine sets but in each step of the construction, we add a random translitional error. Let me call these sets `randomly perturbed self-affine sets`. Below we show an example of fractal percolation set and an example of randomly perturbed self-affine set:

Example of a fractal percolation set Given a probability say $p = 0.8$ and a coin which lands on tail with probability p when it is flipped. We partition (mod 0) the interval $I = [0, 1]$ into 3 sub intrevals $I_k := \left[\frac{k}{3}, \frac{k+1}{3}\right]$, $k = 0, 1, 2$ and we flip the coin independently 3 times for each of these intervals. If the coin lands tail we retain the corresponding interval, otherwise we discard it. Assume we retained intervals I_0 and I_1. The discarded interval I_2 is erased completely. Then we repeat the same process independently in intrevals I_0 and I_1 to obtain some retained second level intervals of length $1/9$. Then inductively, in each retained interval we repeat the same process independently ad infinitum or until we end up with no intervals at all.

Example of a randomly perturbed self-similar set Let $S_i(x) := \frac{1}{3}x + i$ where $i = 0, 1, 3$. Then the interval $I := \left[0, \frac{9}{2}\right]$ satisfies $S_i(I) \subset I$ for all $i \in \mathcal{A} := \{0, 1, 3\}$ The self-similar set which corresponds to the self-similar IFS (Iterated Function System) $\mathcal{S} = \{S_0, S_1, S_3\}$ is

$$\Lambda := \bigcap_{n=1}^{\infty} \bigcup_{(i_1,\dots,i_n)\in\mathcal{A}^n} I_{i_1\dots i_n}, \tag{1}$$

where

$$I_{i_1\dots i_n} := S_{i_1} \circ \cdots \circ S_{i_n}(I) \tag{2}$$

We obtain a randomly perturbed self-similar set from this if we fix a small $\varepsilon > 0$, we write U for the uniform distribution on the interval $(-\varepsilon, \varepsilon)$ and instead of the deterministic interval $I_{i_1\dots i_n}$ we consider its randomly perturbed sister:

$$I^{\mathbf{y}}_{i_1\dots i_n} := (S_{i_1}+y_{i_1})\circ(S_{i_2}+y_{i_1i_2})\circ(S_{i_3}+y_{i_1i_2i_3})\circ\cdots\circ(S_{i_n}+y_{i_1\dots i_n})(I), \quad (3)$$

where $\mathbf{y}=\{y_{\mathbf{j}}\}_{\mathbf{j}\in\bigcup_{n=1}^{\infty}\mathcal{A}^n}$ is a sequence of i.i.d. random variables $y_{\mathbf{j}}\overset{d}{=}U$. The randomly perturbed self-similar set corresponding to the random perturbation $\mathbf{y}$ is

$$\Lambda^{\mathbf{y}} := \bigcap_{n=1}^{\infty}\bigcup_{(i_1,\dots,i_n)\in\mathcal{A}^n} I^{\mathbf{y}}_{i_1\dots i_n}, \quad (4)$$

We place emphasis on the geometric measure theoretical properties (dimension of projections and slices, existence of interior points in the projections, rectifiability) of the random sets under consideration.

Part I

(1) The intorduction of the tools we use from the theory stochastic processes: Branching processes and some elements of Large deviation theory.

(2) Fracatal percolation random sets: the construction, elementary properties and the dimension formula.

(3) Chayes, Chayes, Durrett theorem about the connectivity property of Fractal percolation process [2], [5] .

(4) The orthogonal projections of Fractal percolation sets I. [10], [9].

(5) The orthogonal projections of Fractal percolation sets II. [8], [11].

(6) Frcatal percolation is unrectifiable [1].

(7) Frcatal percolation peocess on Sierpinski carpet and on Menger sponge.

Part II

(1) The definition, dimension and measure of randomly perturbed self-affine sets. The self-affine transversality condition. [4].

(2) Generalized Transversality Condition for dominated triangular C^1 IFS. [3].

(3) The existence of interior points in randomly perturbed self-similar sets.

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