CHAPTER I

BASIC CONCEPTS AND THEOREMS IN THE THEORY OF TRIGONOMETRIC SERIES

§ 1. The concept of a trigonometric series; conjugate series

A *trigonometric series* is the name given to an expression of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \tag{1.1}$$

where a_n and b_n are constants ($n = 0, 1, 2 \ldots$), known as *the coefficients of the series.*†

If such a series converges for all x in $-\infty < x < +\infty$, then it represents a function possessing a period of 2π. Therefore, if a function is to be represented by a trigonometric series, either periodic functions with period 2π are considered or a function is taken which is given in an interval of length 2π and is then expanded periodically, that is, it is required that $f(x + 2\pi) = f(x)$ for any x.

Trigonometric series play a prominent role not only in mathematics itself but also in very many of its applications. But before we discuss this role, we will mention first the connection between trigonometric and power series. If we consider the series

$$\sum_{n=0}^{\infty} c_n z^n, \tag{1.2}$$

where $c_n = a_n - ib_n$, $c_0 = a_0/2$ and we suppose that $z = re^{ix}$, then the series (1.1) is no different from the real part of series (1.2) on the unit circle; the purely imaginary part of the series (1.2) for $z = e^{ix}$ is the series

$$\sum_{n=1}^{\infty}(-b_n \cos nx + a_n \sin nx), \tag{1.3}$$

which is usually called *the series conjugate to series* (1.1).

If it is assumed that the constants c_n are bounded then the series (1.2) represents an analytic function inside a unit circle, that is, for $z = re^{ix}$, where $0 \leqslant r < 1$ and $0 \leqslant x \leqslant 2\pi$; therefore its real and imaginary parts

$$u(r, x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)r^n$$

† The reason why the free term is written as $a_0/2$ will become clear later (see § 4).

and

$$v(r, x) = \sum_{n=1}^{\infty} (- b_n \cos nx + a_n \sin nx) r^n$$

are conjugate harmonic functions; whence is derived the name "conjugate series". The study of the behaviour of conjugate series is no different from an investigation of the behaviour of conjugate harmonic functions on the circle $|z| = 1$.

§ 2. The complex form of a trigonometric series

It is often more convenient to give the trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{2.1}$$

a different form. Thus, from the well-known Euler's identity

$$e^{ix} = \cos x + i \sin x$$

it follows that

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}; \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i},$$

so that we can write series (2.1) in the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \frac{e^{inx} + e^{-inx}}{2} + i b_n \frac{e^{-inx} - e^{inx}}{2} \right),$$

whence, supposing that

$$c_0 = \frac{a_0}{2}, \qquad c_n = \frac{a_n - i b_n}{2}, \qquad c_{-n} = \frac{a_n + i b_n}{2}, \tag{2.2}$$

we see that the series (2.1) takes the form

$$\sum_{n=-\infty}^{n=+\infty} c_n e^{inx}. \tag{2.3}$$

This is the so-called *complex form of the trigonometric series.* The partial sum of series (2.1), that is,

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx),$$

now takes the form

$$S_n(x) = \sum_{k=-n}^{k=+n} c_k e^{ikx}, \tag{2.4}$$

that is, the convergence of series (2.3) must be understood as the tending to the limit of sums of the form (2.4).

Some problems are concerned with trigonometric series of the form (2.3), the coefficients of which are any complex numbers. If it is assumed that the numbers a_n

and b_n in series (2.1) are all real, then from formula (2.2) it is seen that the numbers c_n and c_{-n} are conjugate complex numbers, that is, $c_{-n} = \overline{c_n}$ (the symbol $\bar{a}$ always indicates the number conjugate to a).

§ 3. A brief historical synopsis

The possibility of representing a function by a trigonometric series was first considered by Euler in 1753 in connection with the work by Daniel Bernoulli on "Vibrating Strings" which had appeared at that time.

If a string, fixed at both ends, is disturbed from its state of equilibrium and is allowed to vibrate freely without being given any initial velocity, then Bernoulli affirmed that the position of the string at time t is determined by the formula

$$y = \sum_{p=1}^{\infty} \alpha_p \sin p \frac{\pi x}{l} \cos pkt,$$

where l is the length of the string and k is some coefficient which depends on the density and tension of the string. The coefficients α_p are arbitrary constants and it is possible to choose them so that the initial condition is satisfied, namely, the requirement that initially the string occupies a certain given position.

Euler noticed that this assertion by Bernoulli leads to a paradoxical result, according to the views of mathematicians of that time. Indeed, if $y = f(x)$ is the initial position of the string, then assuming $t = 0$, we should obtain

$$f(x) = \sum_{p=1}^{\infty} \alpha_p \sin p \frac{\pi x}{l},$$

that is, the "arbitrary" function $f(x)$ can be expanded as a sine series. However, Euler and his contemporaries divided the curves into two classes: those that they called "continuous" and the others "geometrical". In contrast to the terminology adopted today, a curve was named "continuous" if y and x were connected by some formula: on the other hand, a geometrical curve was the name given to any curve which could be drawn "free-hand". It is evident from all this that if the curve is given by a formula, then being determinable in some small interval, it is automatically determinable everywhere else†. Therefore they did not doubt that the second category of curves was wider than the first, since they could not consider, for example, a broken line to be "continuous", but merely composed of sections of continuous lines.

If an "arbitrary" function could be expanded as a sine series, i.e. represented by a formula, this would signify that any kind of "geometrical" curve is a "continuous" curve which appeared to be incredible. In particular, D'Alembert noticed that the most natural method of disturbing a string from its state of equilibrium is to take hold of some point on it and pull it upwards, so that it takes up a position represented by two straight lines forming an angle between them. D'Alembert considered that a curve of this nature could not be the sum of a sine series††.

† This property is inherent in analytic functions.

†† For the result of the argument between Euler and D'Alembert concerning the definition of an "arbitrary function", which arose in connection with the solution of the problem of the vibrating string, see the extremely interesting report on "Functions" by N. N. Lusin[4] (it should also appear in Vol. III of the Collected Works of N. N. Lusin).

The problem of what functions can be represented by trigonometric series arose again considerably later in Fourier's researches. In connection with the study of the problem of heat transfer he was confronted with the following problem: let the given function be

$$f(x) = \begin{cases} -1 & \text{in} \quad -\pi < x < 0, \\ 1 & \text{in} \quad 0 < x < \pi. \end{cases}$$

It is required to represent it in the form

$$\sum_{n=1}^{\infty} \alpha_n \sin nx. \tag{3.1}$$

Fourier indicated formulae with the help of which α_n can be determined so that series (3.1) can have $f(x)$ for its sum. In this way, it is a series of form

$$\frac{4}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \cdots + \frac{\sin(2n+1)x}{2n+1} + \cdots\right].$$

Fourier did not prove that the series is bound to converge to the function $f(x)$, but this question was answered in the affirmative by later investigations. In any case it is important that Fourier first solved the problem of how to determine the coefficients of a trigonometric series for it to be able to possess a given function as its sum. It is an entirely different question whether this series does indeed converge and does really possess this function as its sum.

§ 4. Fourier formulae

Let us assume that the function $f(x)$ is not only the sum of a trigonometric series but also that this series converges uniformly in $-\pi \leqslant x \leqslant \pi$; then its coefficients can be determined very easily. This follows simply by multiplying

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

by $\cos kx$ or by $\sin kx$, by integrating it between the limits $-\pi$ to $+\pi$ (which is valid) and noting that

$$\left.\begin{aligned} &\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0, \quad m \neq n, \\ &\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0, \quad m \neq n, \\ &\int_{-\pi}^{\pi} \cos mx \sin nx \, dx = 0, \quad m \neq n \quad \text{and} \quad m = n, \\ &\int_{-\pi}^{\pi} \cos^2 mx \, dx = \int_{-\pi}^{\pi} \sin^2 mx \, dx = \pi. \end{aligned}\right\} \tag{4.1}$$

As a result we obtain†

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx; \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx. \tag{4.2}$$

Formulae (4.2) are called *Fourier formulae*,†† the numbers a_n and b_n are Fourier coefficients and finally the series, the coefficients of which are determined by Fourier formulae derived from the function $f(x)$, is named *the Fourier series* of the function $f(x)$. We will denote it by $\sigma(f)$.

§ 5. The complex form of a Fourier series

If the series representing $f(x)$ is given in a complex form (see § 2)†††, i.e., if we suppose that

$$f(x) = \sum_{n=-\infty}^{n=+\infty} c_n e^{inx}, \tag{5.1}$$

then the coefficients c_n are determined by the formulae

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt \quad (n = 0, \pm 1, \ldots), \tag{5.2}$$

which can be obtained either by starting from equalities (2.2) and substituting the values for a_n and b_n from the Fourier formulae or in a similar manner to that by which the Fourier formulae themselves were produced. Namely, by supposing that

$$f(x) = \sum_{k=-\infty}^{k=+\infty} c_k e^{ikx}. \tag{5.3}$$

where the convergence is uniform, multiplying both sides of equality (5.3) by e^{-inx} and integrating term by term, we find that

$$\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \sum_{k=-\infty}^{k=+\infty} c_k \int_{-\pi}^{\pi} e^{i(k-n)x}dx.$$

But

$$\int_{-\pi}^{\pi} e^{i(k-n)x}dx = \begin{cases} 0, & \text{if } k \neq n, \\ 2\pi, & \text{if } k = n. \end{cases} \tag{5.4}$$

† The free term of the series must be written in the form $a_0/2$ for a_0 to be obtained from a_n when $n = 0$.

†† Strictly speaking, these formulae were already known to Euler, but Fourier began to use them systematically; therefore they are traditionally called Fourier formulae and the corresponding series Fourier series.

††† For references to the text or formulae from the same chapter, the number of the chapter is omitted.

whence

$$\int_{-\pi}^{\pi} f(x) e^{-inx} dx = 2\pi c_n,$$

which proves the validity of formula (5.2).

The numbers c_n are called *the complex Fourier coefficients of the function* $f(x)$.

§ 6. Problems in the theory of Fourier series; Fourier–Lebesgue series

In §§ 4 and 5 we have solved only the problem of how the coefficients of a trigonometric series should be determined if we know that it converges uniformly to some function $f(x)$. It was shown that in this case the series possesses coefficients determinable by Fourier formulae, that is, it is a Fourier series of $f(x)$.

However, for the function to be the sum of an uniformly convergent series of continuous functions, it is necessary that it be continuous. Therefore, it could appear that if it is desired to represent a function by a Fourier series, we must confine ourselves to the case when it is continuous. We will see that in fact the theory of Fourier series embraces a very much wider class of functions. But first of all we must define more exactly what we understand by Fourier series.

Integrals figure in Fourier formulae. We know that the concept of an integral, starting with Cauchy, has widened, so that an increasingly large class of integrable functions has developed. In this book we will always understand by the class of "integrable functions" those integrable according to Lebesgue. These functions, as is known, are called summable; the series set up for them are named the Fourier–Lebesgue series. For brevity's sake we shall simply say "Fourier series" but at the same time realise that the series being considered are always summable.

Let $f(x)$ be summable in $[-\pi, \pi]$. Then it is always possible to determine for it the numbers a_n and b_n from Fourier formulae and to set up a series which we will name the Fourier series for this function and write

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{6.1}$$

or

$$\sigma(f) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \tag{6.2}$$

The sign $\sim$ indicates that we established this series in a purely formal manner, starting from $f(x)$ and using Fourier formulae, but we know nothing of the convergence of this series. A whole succession of problems arises: should the Fourier series converge (in the whole interval $[-\pi, \pi]$ or at a given point or in a certain set) and if so, does it converge to the function $f(x)$ or not? In which cases will the convergence be absolute, when will it be uniform? What can be said of divergent Fourier series (is it possible to use them in any way for assessing functions?). These and many other problems will be discussed in later chapters of this book.

It should also be mentioned that there are cases when the trigonometric series is given by its coefficients but we do not know whether it is a Fourier series of a certain function or not. This is one of the very interesting but difficult problems of the theory of trigonometric series.

§ 7. Expansion into a trigonometric series of a function with period $2l$

Up until now we have considered the expansion into a trigonometric series of a function with period 2π. If the function $f(x)$ has a period $2l$, where l is some real number, then performing a change of variable,

$$x = \frac{lt}{\pi},$$

we obtain the function

$$\varphi(t) = f\left(\frac{lt}{\pi}\right),$$

which will also possess a period 2π.

If we find its Fourier series

$$\varphi(t) \sim \frac{a_0}{2} + \sum (a_n \cos nt + b_n \sin nt),$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \varphi(t)\cos nt\,dt; \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \varphi(t)\sin nt\,dt,$$

then, reverting again to the variable x, we obtain

$$\left.\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{lt}{\pi}\right)\cos nt\,dt = \frac{1}{l}\int_{-l}^{l} f(x)\cos n\frac{\pi}{l}x\,dx, \quad n = 0, 1, \ldots, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{lt}{\pi}\right)\sin nt\,dt = \frac{1}{l}\int_{-l}^{l} f(x)\sin n\frac{\pi}{l}x\,dx, \quad n = 1, 2, \ldots, \end{aligned}\right\} \tag{7.1}$$

and therefore the function $f(x)$ will correspond to the series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos n\frac{\pi}{l}x + b_n \sin n\frac{\pi}{l}x\right), \tag{7.2}$$

where the numbers a_n and b_n are determined by the formulae (7.1).

Everything that will be said later concerning the convergence of normal trigonometric series is completely applicable to series of the form (7.2).

Finally we consider the case when the function $f(x)$ is not periodic. If it is defined in a certain interval $[a, b]$ where $-\pi < a < b < \pi$ (Fig. 4) and is summable in it, then it is possible to expand it into a trigonometric series thus: construct a function $\varphi(x)$ coinciding with $f(x)$ in $[a, b]$ and defined in $(-\pi, a)$ and (b, π) as desired, provided that it is summable. Then assuming that $\varphi(x + 2\pi) = \varphi(x)$ we expand $\varphi(x)$ into a Fourier series. We assume that this series converges to $\varphi(x)$ at a certain point x, $a < x < b$; this means that its sum at this point will equal $f(x)$.

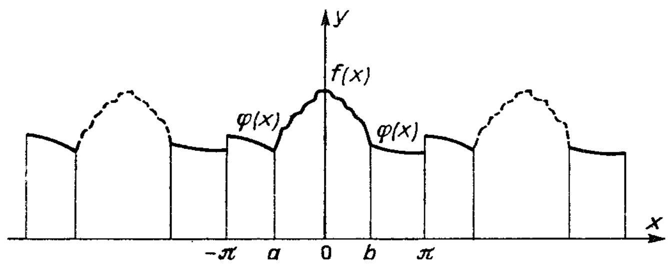

FIG. 4

It is clear that on extending $f(x)$ by various means outside the limits (a, b), we will obtain various functions $\varphi(x)$. However, it will be proved subsequently (see § 33) that the Fourier series of all these functions will behave identically, that is, if one of them converges to $f(x)$ at a given point, then all the others will also converge likewise.

§ 8. Fourier series for even and odd functions

If $f(x)$ is even, i.e. $f(-x) = f(x)$ and $g(x)$ is odd, i.e. $g(-x) = -g(x)$, then $f(x)\,g(x)$ is evidently odd; on the other hand, if $f(x)$ and $g(x)$ are both even or both odd, then $f(x)\,g(x)$ is even.

It can be concluded immediately from this simple statement that for any even function the Fourier series contains cosines alone and for any odd function sines alone. Indeed, for any odd function $\varphi(x)$ and for any $a > 0$ we have

$$\int_{-a}^{a} \varphi(x)dx = 0,$$

and therefore for even $f(x)$ we have

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0 \quad (n = 1, 2, 3, \ldots),$$

and for odd $f(x)$ we have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0, \quad (n = 0, 1, \ldots).$$

Moreover, for any even $\varphi(x)$ and for any $a > 0$ we have

$$\int_{-a}^{a} \varphi(x)\,dx = 2\int_{0}^{a} \varphi(x)\,dx.$$

Therefore, in conclusion: if $f(x)$ is even, then

$$\sigma(f) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx,$$

where

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx;$$

if $f(x)$ is odd, then

$$\sigma(f) = \sum_{n=1}^{\infty} b_n \sin nx,$$

where

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx.$$

§ 9. Fourier series with respect to the orthogonal system

When we set ourselves the task of defining the coefficients of a trigonometric series so that it converges to a given function $f(x)$ we only considered a particular case of an extremely general problem. In order to formulate this problem we introduce the concept of an orthogonal system.

A system of functions $\varphi_n(x) \in L^2(a, b)$ $(n = 1, 2, \dots)$ is said to be *orthogonal* in the interval $[a, b]$, if

$$\left.\begin{aligned} &\int_a^b \varphi_m(x)\varphi_n(x)dx = 0 \quad m \neq n; \quad m = 1, 2, \dots; \quad n = 1, 2, \dots, \\ &\int_a^b \varphi_n^2(x)dx \neq 0 \qquad n = 1, 2, \dots. \end{aligned}\right\} \tag{9.1}$$

The relationships (4.1) are simply proof of the orthogonality of the trigonometric system

$$1, \cos x, \sin x, \dots, \cos nx, \sin nx, \dots$$

in the interval $[-\pi, \pi]$.

The orthogonal system is said to be *normal*, if

$$\int_a^b \varphi_n^2(x)dx = 1 \quad (n = 1, 2, \dots).$$

Rademacher's system[1] can serve as an example of a normal orthogonal system; it is set up thus: the interval [0,1] is divided into 2^n equal intervals and the function $r_n(x)$ is assumed to equal $+1$ in the first, third, ..., $(2^n - 1)$th interval and to equal -1 in the second, fourth, ..., 2^n th interval (i.e. it assumes alternately the values $+1$ and -1) and at the end points of the intervals it is considered to equal zero. This holds for all values of n $(n = 1, 2, \dots)$. The orthogonality of the system $\{r_n(x)\}$ obtained in the interval [0,1] follows from the fact that if $m \neq n$ (let $m < n$), then the function $r_n(x)$ in every interval when $r_m(x)$ is constant takes the value $+1$ just as many times as the value -1 and the lengths of the intervals in which it is constant are all equal. Thus we are satisfied that

$$\int_0^1 r_m(x) r_n(x)\, dx = 0 \quad (m \neq n).$$

Since for any n we have $r_n^2(x) = 1$ everywhere, apart from a finite number of points, then the system $\{r_n(x)\}$ is normal†.

Later whilst studying the properties of trigonometric series Rademacher's system will prove very useful.

A trigonometric system is not normal but can be made normal, if the first function is multiplied by $1/\sqrt{2\pi}$ and all the other functions by $1/\sqrt{\pi}$, that is, the system

$$\frac{1}{\sqrt{2\pi}}, \quad \frac{\cos x}{\sqrt{\pi}}, \quad \frac{\sin x}{\sqrt{\pi}}, \ldots, \quad \frac{\cos nx}{\sqrt{\pi}}, \quad \frac{\sin nx}{\sqrt{\pi}}, \quad \ldots$$

is already a normal orthogonal system.

We will not consider the question why the study of orthogonal systems is extremely interesting and important. Specialized books are devoted to this question. Here we shall merely show that a whole series of theorems concerning the theory of trigonometric series can be obtained extremely easily, starting from very general results relating to the so-called orthogonal series.

A series of the form

$$\sum_{n=1}^{\infty} c_n \varphi_n(x), \tag{9.2}$$

where c_n are constant coefficients and $\{\varphi_n(x)\}$ is a given orthogonal system of functions, is called *a series with respect to the orthogonal system* $\{\varphi_n(x)\}$ or briefly, *an orthogonal series.*

In the same way as we described how to find the coefficients of a trigonometric series if we know that it converges to a certain function $f(x)$, we can discuss how to determine the coefficients c_n, if we know that

$$f(x) = \sum_{n=1}^{\infty} c_n \varphi_n(x). \tag{9.3}$$

We again assume that the series converges uniformly. We suppose that the system $\{\varphi_n(x)\}$ is orthogonal and normal in (a, b). Then multiplying both sides of equality (9.3) by $\varphi_m(x)$ and integrating between the limits from a to b we find††

$$\int_a^b f(x)\varphi_m(x)\, dx = c_m \int_a^b \varphi_m^2(x)\, dx = c_m,$$

† The reader can find more detail of the properties of Rademacher's system in Kaczmarz and Steinhaus's book, ref. A 12.

†† Here the functions $\varphi_n(x)$ and $f(x)$ are supposed to be such that the integrals (9.4) have meaning.

i.e.

$$c_m = \int_a^b f(x)\varphi_m(x)dx \quad (m = 1, 2, \ldots). \tag{9.4}$$

These formulae are also called *Fourier formulae* and if for some functions $f(x)$ the numbers c_n are found from the formulae (9.4) and the series (9.2) is formed from them, then it is named the *Fourier series for the function $f(x)$ with respect to the orthogonal system $\{\varphi_n(x)\}$*.

Here, as in the case of the trigonometric system, the hypothesis of uniform convergence of the system was extremely limiting. We can consider a Fourier series for the function $f(x)$ with the single assumption that the integrals (9.4) have meaning and then write

$$f(x) \sim \sum_{n=1}^{\infty} c_n\varphi_n(x).$$

Just as in the theory of trigonometric series, the question arises of the convergence of the Fourier series and to what extent it characterizes the function $f(x)$.

It is, above all, clear that for the Fourier series to be able to define to any extent the properties of a function, it is necessary that there should not be identical Fourier series for two different functions. To explain the problem when this does occur, we must first study the concept of the completeness of an orthogonal system. The problem will be discussed in § 10. Here we shall just describe what we understand by an orthogonal system in the case when the functions $\varphi_n(x)$ are complex.

If the functions $\varphi_n(x)$ are complex functions of the real variable x, then they are said to be *orthogonal* when

$$\int_a^b \varphi_m(x)\bar{\varphi}_n(x)dx = 0 \quad (m \neq n) \tag{9.5}$$

and

$$\int_a^b |\varphi_n(x)|^2 dx \neq 0, \quad (n = 1, 2, \ldots). \tag{9.6}$$

The system is *normal* if

$$\int_a^b |\varphi_n(x)|^2 dx = 1 \quad (n = 1, 2, \ldots).$$

In the case of complex functions the Fourier formulae take the form

$$c_n = \int_a^b f(x)\bar{\varphi}_n(x)dx \tag{9.7}$$

for normal systems and

$$c_n = \frac{\int_a^b f(x)\bar{\varphi}_n(x)dx}{\int_a^b |\varphi_n(x)|^2 dx}$$

for non-normal systems.

An important example of an orthogonal system of complex functions is the system $\{e^{inx}\}$ $(n = 0, \pm 1, \pm 2, \ldots)$; it is orthogonal over any interval of length 2π (see § 5). If the multiplier $1/\sqrt{2\pi}$ is introduced, i.e., if the system

$$\left\{\frac{1}{\sqrt{2\pi}}\, e^{inx}\right\} \quad (n = 0, \pm 1, \ldots),$$

is considered, then it is also normal.

§ 10. Completeness of an orthogonal system

We now introduce the following important definition.†

DEFINITION. The system of functions $\{\varphi_n(x)\}$, defined in some interval $[a, b]$, is said to be complete in $L^p\,[a, b]$ $(p \geqslant 1)$ (or in $C\,[a, b]$) if there does not exist a single function $f(x) \in L^p\,[a, b]$ (or $f(x) \in C\,[a, b]$), which is orthogonal to all the functions of this system, unless $f(x) = 0$ almost everywhere in $[a, b]$ (for the case of the space C, everywhere in $[a, b]$).

In other words, for a complete system of the equalities

$$\int_a^b f(x)\varphi_n(x)dx = 0 \quad (n = 1, 2, \ldots) \tag{10.1}$$

and for $f(x) \in L^p\,[a, b]$ it should follow that $f(x) = 0$ almost everywhere in $[a, b]$ (similarly for space C, but the word "everywhere" should be substituted for the words "almost everywhere").

For the integrals occurring in (10.1) to have meaning for any $f(x) \in L\,[a, b]$, it is necessary and sufficient for all $\varphi_n(x)$ to be bounded in $[a, b]$; if $f(x) \in L^p\,[a, b]$, then it is necessary and sufficient for $\varphi_n(x) \in L^q\,[a, b]$ $(n = 1, 2, \ldots)$ where $1/p + 1/q = 1$ (see Introductory Material, § 9 and Appendix, § 3), finally for $f(x) \in C$ from the functions $\varphi_n(x)$ only summability is required.

The concept of completeness is introduced without assuming the orthogonality of the system $\{\varphi_n(x)\}$ but we will be interested in the case when it is orthogonal.

If the functions $\varphi_n(x)$ are complex, then the definition holds, only instead of equations (10.1) we must write

$$\int_a^b f(x)\bar{\varphi}_n(x)dx = 0 \quad (n = 1, 2, \ldots).$$

If the two functions $f(x) \in L^p\,[a, b]$ and $g(x) \in L^p\,[a, b]$ are different in a set of measure greater than zero, then they cannot possess identical Fourier series with respect to a system of functions $\{\varphi_n(x)\}$ complete in $L^p\,[a, b]$ (at $p \geqslant 1$). Indeed, if this were the case, then the difference $\psi(x) = f(x) - g(x)$ would be functions belonging to $L^p\,[a, b]$ and orthogonal to all $\{\varphi_n(x)\}$, whilst the condition $\psi(x) = 0$ almost everywhere in $[a, b]$ is not fulfilled and this contradicts the definition of completeness of the system.

† For all the notation used here reference should be made to the Notation (p. xxiii).

§ 11. Completeness of the trigonometric system in the space L

We shall prove that the trigonometrical system is complete in the space $L(-\pi, \pi)$, i.e., we shall demonstrate that two summable functions possess identical trigonometrical Fourier series only in the case when they coincide almost everywhere in $(-\pi, \pi)$.

For this we prove first of all that if the completeness of the trigonometric system is already known in C, then we can immediately obtain from it its completeness in L.

In fact, we assume that $f(x) \in L$ and

$$\left.\begin{aligned} \int_{-\pi}^{\pi} f(x) \cos nx \, dx &= 0 \quad (n = 0, 1, \ldots), \\ \int_{-\pi}^{\pi} f(x) \sin nx \, dx &= 0 \quad (n = 1, 2, \ldots). \end{aligned}\right\} \tag{11.1}$$

Then denoting the Fourier coefficients of $f(x)$ by a_n and b_n we have

$$a_n = 0 \quad (n = 0, 1, \ldots),$$

$$b_n = 0 \quad (n = 1, 2, \ldots).$$

Let us consider the function

$$F(x) = \int_{-\pi}^{x} f(t) \, dt$$

in

$$-\pi \leqslant x \leqslant \pi$$

and

$$F(x + 2\pi) = F(x).$$

It is clear that $F(\pi) = \pi a_0 = 0$ and $F(-\pi) = 0$, consequently $F(x)$ is continuous not only in $[-\pi, \pi]$ but also along the whole straight line $-\infty < x < +\infty$. We find its Fourier coefficients A_n and B_n by integrating by parts, so that

$$A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \cos nx \, dx = \frac{1}{n\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = 0.$$

(due to (11.1)) and similarly

$$B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \sin nx \, dx = \frac{1}{n\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = 0 \quad (n = 1, 2, \ldots).$$

Thus, all the Fourier coefficients for $F(x)$ apart from A_0 should be equal to zero. Since $F(x)$ is continuous, then supposing $\Phi(x) = F(x) - A_0/2$, we see that $\Phi(x)$ is continuous and all its Fourier coefficients equal zero, i.e., it is orthogonal to all the functions of the trigonometric system. But we have already assumed that the trigonometric system is complete in C. This means that $\Phi(x) \equiv 0$ and therefore $F(x) = A_0/2$

= const. But since $F'(x) = f(x)$ almost everywhere, then $f(x) = 0$ almost everywhere and this is what was required to be proved.

We will now prove the completeness of the system in C.

We have defined (Introductory Material, § 22) a trigonometric polynomial as any expression of the form

$$T_n(x) = \alpha_0 + \sum_{k=1}^{n} (\alpha_k \cos kx + \beta_k \sin kx). \tag{11.2}$$

It is clear that if $f(x)$ is orthogonal to all the functions of the trigonometric system, then it is orthogonal also to any trigonometric polynomial, i.e. for any $T_n(x)$

$$\int_{-\pi}^{\pi} f(x) T_n(x) dx = 0. \tag{11.3}$$

We will show that if $f(x)$ is continuous but not identically equal to zero, then a trigonometric polynomial $T_n(x)$ can be chosen such that the integral on the left-hand side of equation (11.3) is positive; then it becomes clear that it is only possible to avoid the contradiction if it is assumed that $f(x) \equiv 0$.

Thus, let $f(x) \not\equiv 0$; then a point ξ can be found such that $f(\xi) = c \neq 0$. It can be assumed that $c > 0$, without altering the whole argument (since in the opposite case, it would be sufficient to show that $-f(x) \equiv 0$). It can also be assumed that $\xi = 0$, since if we are able for the functions $\varphi(x)$, of which $\varphi(0) > 0$, to find a polynomial $T_n^*(x)$ for which

$$\int_{-\pi}^{\pi} \varphi(x) T_n(x) dx > 0,$$

then, supposing $\varphi(x) = f(\xi + x)$ and $T_n(x) = T_n^*(x - \xi)$, we see that

$$\int_{-\pi}^{\pi} f(t) T_n(t) dt = \int_{-\pi}^{\pi} f(\xi + x) T_n(\xi + x) dt = \int_{-\pi}^{\pi} \varphi(x) T_n^*(x) dx > 0.$$

Thus, it remains to prove that if $f(0) = c > 0$, it is possible to find a polynomial $T_n(x)$ for which

$$\int_{-\pi}^{\pi} f(x) T_n(x) dx > 0. \tag{11.4}$$

But if $f(0) = c > 0$, then because of the continuity of $f(x)$ it is possible to find an interval $(-\delta, +\delta)$ where $f(x) \geqslant c/2$. We have

$$\int_{-\pi}^{\pi} f(x) T_n(x) dx = \int_{-\delta}^{\delta} f(x) T_n(x) dx + \int_{-\pi}^{-\delta} f(x) T_n(x) dx + \int_{\delta}^{\pi} f(x) T_n(x) dx.$$

Since $f(x)$ is continuous, then it is bounded, i.e.

$$|f(x)| \leqslant M \quad -\pi \leqslant x \leqslant \pi, \tag{11.5}$$

where M is a constant.

Let $A > 0$ be given. We will assume that $T_n(x)$ can be chosen such that the following conditions are satisfied

$$T_n(x) \geqslant 1 \quad \text{in} \quad (-\delta, \delta), \tag{11.6}$$

$$\int_{-\pi}^{\pi} T_n(x)\,dx > A \tag{11.7}$$

and

$$|T_n(x)| \leqslant 1 \quad \text{in} \quad (-\pi, \delta) \quad \text{and} \quad (\delta, \pi). \tag{11.8}$$

Let us take $A > 4M\pi/c$, where M is given by condition (11.5). Then

$$\int_{-\pi}^{\pi} f(x)T_n(x)\,dx > \frac{cA}{2} - M \cdot 2\pi > 0$$

and this signifies that (11.4) occurs and now the proof will be concluded.

So, it remains to choose a trigonometric polynomial $T_n(x)$ such that the conditions (11.6), (11.7) and (11.8) are satisfied.

To find this polynomial we note that if

$$T(x) = 1 + \cos x - \cos\delta,$$

then $T(x) \geqslant 1$ in $(-\delta, \delta)$ and $|T(x)| \leqslant 1$ outside $(-\delta, \delta)$, and therefore for

$$T_n(x) = [T(x)]^n$$

we also have

$$|T_n(x)| \leqslant 1 \quad \text{outside} \quad (-\delta, \delta) \quad \text{and} \quad T_n(x) \geqslant 1 \quad \text{in} \quad (-\delta, \delta).$$

Moreover, in $(-\delta/2, \delta/2)$ we have

$$T(x) > 1 + \cos\frac{\delta}{2} - \cos\delta = q > 1,$$

and therefore

$$\int_{-\delta}^{\delta} T_n(x)\,dx > \int_{-\delta/2}^{\delta/2} T_n(x)\,dx > q^n\delta \to \infty$$

as $n \to \infty$, which means that for any A, by choosing n sufficiently large, the inequality (11.7) can be fulfilled.

It remains to prove that $T_n(x)$ is a trigonometric polynomial. But since $T(x) = \cos x + c$, where c is a constant, then $[T(x)]^n$ is a trigonometric polynomial for any n (see Introductory Material, § 22).

Thus, our theorem is completely proved. From the very definition of completeness of the system in the space L^p it follows that if $p' > p$, then the completeness in L^p implies completeness in $L^{p'}$. In particular, the trigonometric system which is complete in L (§ 11) will be complete also in L^p for any $p > 1$.

§ 12. Uniformly convergent Fourier series

From the completeness of the trigonometric system in C, the following simple but important conclusion can be drawn:

THEOREM. *If the Fourier series for a continuous function $f(x)$ converges uniformly, then the sum of this series coincides with $f(x)$.*

Indeed, let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

where $f(x)$ is continuous, and the series on the right-hand side converges uniformly in $[-\pi, \pi]$. We will denote its sum by $S(x)$. It is clear that $S(x)$ is continuous. But we have seen (see § 4) that if $S(x)$ is the sum of an uniformly convergent trigonometric series, then its coefficients a_n and b_n are obtained from $S(x)$ by means of the Fourier formulae. On the other hand, it is conditional that a_n and b_n are obtained from $f(x)$ by means of the Fourier formulae. Thence it follows that $S(x)$ and $f(x)$ possess identical Fourier coefficients. Therefore, because of the completeness of the trigonometric system in C, they should coincide identically.

Later (see § 48) we will show that in this theorem the requirement of uniform convergence can be discarded and it can be affirmed that if $f(x)$ is continuous, then at any point where its Fourier series converges, it converges to $f(x)$.

At the present moment, as we are referring to uniformly convergent series, it is appropriate to prove one lemma, which will be used frequently later.

LEMMA. *Let the trigonometric series $a_0/2 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ possess a sub-sequence of partial sums, converging uniformly to some function $f(x)$. Then this series is its Fourier series* (in particular, this statement is more accurate, when the series itself converges uniformly to $f(x)$).

Indeed, let $S_{n_k}(x)$ $(k = 1, 2, \ldots)$ converge uniformly to $f(x)$. Then, the more so

$$\int_{-\pi}^{\pi} |f(x) - S_{n_k}(x)|\, dx \to 0 \quad \text{as} \quad k \to \infty.$$

Hence for any m we have

$$\int_{-\pi}^{\pi} [f(x) - S_{n_k}(x)] \cos mx\, dx \to 0 \quad \text{as} \quad k \to \infty,$$

$$\int_{-\pi}^{\pi} [f(x) - S_{n_k}(x)] \sin mx\, dx \to 0 \quad \text{as} \quad k \to \infty,$$

i.e.

$$\lim_{k\to\infty} \int_{-\pi}^{\pi} S_{n_k}(x) \cos mx\, dx = \int_{-\pi}^{\pi} f(x) \cos mx\, dx$$

and similarly for $\sin mx$. But because of the orthogonality of the trigonometric system, if $n_k \geqslant m$, then we have

$$\int_{-\pi}^{\pi} S_{n_k}(x) \cos mx\, dx = a_m \int_{-\pi}^{\pi} \cos^2 mx\, dx = \pi a_m,$$

and therefore

$$\lim_{k\to\infty} \int_{-\pi}^{\pi} S_{n_k}(x) \cos mx\, dx = \pi a_m$$

which means

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx\, dx$$

and similarly for b_m. Thus the lemma is proved.

§ 13. The minimum property of the partial sums of a Fourier series; Bessel's inequality

Let us now return to the general case, i.e., to considering the Fourier series with respect to any orthogonal system. We will refer to orthogonal systems complete in L^2, since they possess a number of important properties which we will proceed to study.

Let $\{\varphi_n(x)\}$ be complete in L^2 $[a, b]$ and orthogonal and normal in this interval. We set ourselves the following problem: given a function $f(x) \in L^2$, we take n functions of the system $\{\varphi_n(x)\}$ and consider all possible expressions of the form $\sum_{k=1}^{n} \alpha_k \varphi_k(x)$, which are known as polynomials of the nth order with respect to the system $\{\varphi_n(x)\}$. We want to know how to choose the constants $\alpha_1, \alpha_2, \ldots, \alpha_n$ so that the polynomial $\sum_{k=1}^{n} \alpha_k \varphi_k(x)$ gives the best approximation for $f(x)$ in the metric space L^2, i.e. for the norm of the difference

$$\| f(x) - \sum_{k=1}^{n} \alpha_k \varphi_k(x) \| L^2$$

to be a minimum. We will prove a theorem.

THEOREM. *Of all the polynomials of the n-th order with respect to a normal orthogonal system $\{\varphi_n(x)\}$, the best approximation in the metric space L^2 for $f(x) \in L^2$ is given by the n-th partial sum of its Fourier series with respect to this system.*

In order to verify this theorem which we will prove generally by assuming that $\varphi_n(x)$ is complex, we write, using the identity $|A^2| = A \cdot \bar{A}$:

$$\left\| f(x) - \sum_{k=1}^{n} \alpha_k \varphi_k(x) \right\|_{L^2}^2 = \int_a^b \left| f(x) - \sum_{k=1}^{n} \alpha_k \varphi_k(x) \right|^2 dx$$

$$= \int_a^b \left[f(x) - \sum_{k=1}^{n} \alpha_k \varphi_k(x) \right] \left[\bar{f}(x) - \sum_{k=1}^{n} \bar{\alpha}_k \bar{\varphi}_k(x) \right] dx$$

$$= \int_a^b |f(x)|^2 dx - \sum_{k=1}^{n} \alpha_k \int_a^b \bar{f}(x) \varphi_k(x) dx - \sum_{k=1}^{n} \bar{\alpha}_k \int_a^b f(x) \bar{\varphi}_k(x) dx$$

$$+ \sum_{k=1}^{n} \sum_{j=1}^{n} \alpha_k \bar{\alpha}_j \int_a^b \varphi_k(x) \bar{\varphi}_j(x) dx;$$

and since

$$\int_a^b \varphi_k(x)\bar{\varphi}_j(x)dx = 0 \quad \text{for} \quad k \neq j,$$

$$\int_a^b |\varphi_k(x)|^2 dx = 1 \quad \text{for} \quad k = 1, 2, \ldots,$$

then

$$\int_a^b |f(x) - \sum_{k=1}^n \alpha_k \varphi_k(x)|^2 dx = \int_a^b |f(x)|^2 dx - \sum_{k=1}^n \alpha_k \bar{c}_k - \sum_{k=1}^n \bar{\alpha}_k c_k + \sum_{k=1}^n |\alpha_k|^2,$$

where c_k are the Fourier coefficients of the function $f(x)$.

In other words $\left(\text{adding and subtracting } \sum_{k=1}^n |c_k|^2\right)$,

$$\left\| f(x) - \sum_{k=1}^n \alpha_k \varphi_k(x) \right\|_{L^2}^2 = \|f\|_{L^2}^2 + \sum_{k=1}^n |c_k - \alpha_k|^2 - \sum_{k=1}^n |c_k|^2. \tag{13.1}$$

It is clear that the right-hand side of (13.1) will be a minimum when and only when

$$\alpha_k = c_k \quad (k = 1, 2, \ldots, n),$$

and the theorem is proved.

Substituting the numbers c_k in (13.1) instead of α_k, we obtain as a result

$$\left\| f(x) - \sum_{k=1}^n c_k \varphi_k(x) \right\|_{L^2}^2 = \|f\|_{L^2}^2 - \sum_{k=1}^n |c_k|^2. \tag{13.2}$$

Since the left-hand side of equation (13.2) is non-negative, then the right-hand side is also non-negative and therefore

$$\sum_{k=1}^n |c_k|^2 \leqslant \|f\|_{L^2}^2.$$

This inequality is true for any n and therefore the series $\sum_{k=1}^\infty |c_k|^2$ converges and

$$\sum_{k=1}^\infty |c_k|^2 \leqslant \|f\|_{L^2}^2. \tag{13.3}$$

The inequality (13.3) is called *Bessel's inequality*. It holds for any normal orthogonal system and for any $f(x) \in L^2$.

§ 14. Convergence of a Fourier series in the metric space L^2

An important theorem can be obtained easily from Bessel's inequality.

THEOREM. *For any function with an integrable square, the Fourier series with respect to any normal orthogonal system converges in the metric space L^2.*

In order to prove this assertion, we recall (see Introductory Material, § 21) that for convergence of the sequence $f_n(x)$ in the metric space L^2 it is necessary and sufficient that for any ε it is possible to find N such that

$$\|f_n(x) - f_m(x)\|_{L^2} < \varepsilon \quad \text{for} \quad n \geqslant N \quad \text{and} \quad m \geqslant N.$$

We will show that this criterion is fulfilled if the partial sums $S_n(x)$ of the Fourier series for $f(x) \in L^2$ play the role of the functions $f_n(x)$.

We have for any integer n and $p \geqslant 1$

$$\| S_{n+p}(x) - S_n(x) \|_{L^2}^2 = \left\| \sum_{k=n+1}^{n+p} c_k \varphi_k(x) \right\|_{L^2}^2$$

$$= \int_a^b \left| \sum_{k=n+1}^{n+p} c_k \varphi_k(x) \right|^2 dx = \sum_{k=n+1}^{n+p} |c_k|^2,$$

since the system $\{\varphi_n(x)\}$ is orthogonal and normal. But by virtue of Bessel's inequality we know that if $f(x) \in L^2$, then $\sum_{k=1}^{\infty} |c_k|^2 < +\infty$, and therefore for any $\varepsilon > 0$, it is possible to find N such that $\sum_{n+1}^{n+p} |c_k|^2 < \varepsilon$ for $n \geqslant N$ and then

$$\| S_{n+p}(x) - S_n(x) \|_{L^2} < \varepsilon,$$

which concludes the proof of the convergence of the Fourier series for $f(x) \in L^2$.

However, it should be noted that only the convergence of the Fourier series in the metric space L^2 was proved. It is not evident from this that the sum in the sense of the metric space L^2 of this series should be equal to the function $f(x)$. This in fact is not always the case. The question whether the Fourier series in the metric space L^2 does converge to a given function is linked with the question of the so-called closure of the orthogonal system in the metric space L^2. We will now start discussing this question.

§ 15. Concept of the closure of the system. Relationship between closure and completeness

It is said that the system of functions $\{\varphi_n(x)\}$ is *closed in the space* C in $[a, b]$ or in $L^p (p \geqslant 1)$ in $[a, b]$, if it is possible to represent any function $f(x) \in C$ (or $f(x) \in L^p$) in this space to a given degree of accuracy in the form of a polynomial with respect to the system $\{\varphi_n(x)\}$.

Re-stating this more precisely, the system $\{\varphi_n(x)\}$ is closed in C (or in L^p) if for any $f(x) \in C$ (or $f(x) \in L^p$) and for any $\varepsilon > 0$ it is possible to choose the numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$, so that

$$|f(x) - \sum_{k=1}^{n} \alpha_k \varphi_k(x)| < \varepsilon \quad \text{at} \quad a \leqslant x \leqslant b$$

or

$$\left\| f(x) - \sum_{k=1}^{n} \alpha_k \varphi_k(x) \right\|_{L^p} < \varepsilon.$$

We will now formulate without proof two theorems referring to the connection between closed and complete systems, namely: if $1/p + 1/q = 1$, then every system closed in $L^p (p > 1)$ (or in C) is complete in L^q (or in L). Conversely, every system complete in $L^p (p > 1)$ is closed in L^q.†

† The proof of these theorems can be found, for example, in Kaczmarz and Steinhaus's book, ref. A 12.

We will consider in more detail only the most important case, when $p = 2$. In this case $q = 2$ and the formulation of our theorem leads to:

THEOREM. *In the space L^2 the completeness and closure of a system are equivalent, i.e. every complete system is closed and vice versa.*

This statement can be proved for any systems consisting of functions occurring in L^2. But we shall confine ourselves to a consideration of the case when the given system is orthogonal. Moreover, since neither the closure nor the completeness of a system can disappear or appear, if we multiply all the functions of the system by any constants, then the system can be assumed to be normal.

Thus, let $\{\varphi_n(x)\}$ be a normal orthogonal system in the interval $[a, b]$. We have seen in § 14 that for any $f(x) \in L^2\ [a, b]$ its Fourier series with respect to the system $\{\varphi_n(x)\}$ converges in the metric space L^2. We will denote its sum by $F(x)$, then

$$F(x) = \sum_{k=1}^{\infty} c_k \varphi_k(x), \tag{15.1}$$

where the equal sign is understood to mean convergence in the metric L^2.

We will prove that the numbers c_n are the Fourier coefficients of the functions $F(x)$. In fact, multiplying both sides of equality (15.1) by $\bar{\varphi}_n(x)$ and integrating (this is valid according to Riesz's theorem, see Introductory Material, § 21), we have

$$\int_a^b F(x)\bar{\varphi}_n(x)\,dx = \sum_{k=1}^{\infty} c_k \int_a^b \varphi_k(x)\bar{\varphi}_n(x)\,dx. \tag{15.2}$$

Because of the orthogonality and normality of $\{\varphi_n(x)\}$, we find that

$$c_n = \int_a^b F(x)\bar{\varphi}_n(x)\,dx.$$

Hence we conclude that all the Fourier coefficients of the functions $f(x)$ and $F(x)$ are identical. If it is assumed that the system $\{\varphi_n(x)\}$ is complete, then this is possible only in the case when $f(x) = F(x)$ almost everywhere and therefore we obtain

$$f(x) = \sum_{k=1}^{n} c_k \varphi_k(x).$$

Here the equality sign is again understood in the sense of convergence in L^2. Therefore

$$\left\| f(x) - \sum_{k=1}^{n} c_k \varphi_k(x) \right\|_{L^2} \to 0$$

as $n \to \infty$, i.e. for any $\varepsilon > 0$ it is possible to find N such that

$$\left\| f(x) - \sum_{k=1}^{n} c_k \varphi_k(x) \right\|_{L^2} < \varepsilon.$$

But $f(x)$ was any function of L^2. Therefore, in agreement with the definition of closure, we see that $\{\varphi_n(x)\}$ is closed in L^2.

Thus, we have proved that the completeness of a system in L^2 implies its closure in L^2. The converse is very easily proved.

Let $\{\varphi_n(x)\}$ be a closed system in L^2 and $f(x)$ any function of L^2. Then for any $\varepsilon > 0$ it is possible to choose numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$\left\| f(x) - \sum_{k=1}^{n} \alpha_k \varphi_k(x) \right\| < \varepsilon.$$

But it was proved (see § 13) that of all the polynomials of order n with respect to the system $\{\varphi_n(x)\}$ the best approximation to $f(x)$ in the metric L^2 is given by the polynomial $\sum_{k=1}^{n} c_k \varphi_k(x)$, the coefficients of which are the Fourier coefficients of $f(x)$. Therefore

$$\left\| f(x) - \sum_{k=1}^{n} c_k \varphi_k(x) \right\| \leqslant \left\| f(x) - \sum_{k=1}^{n} \alpha_k \varphi_k(x) \right\| < \varepsilon.$$

But since we know (see (13.2)) that

$$\left\| f(x) - \sum_{k=1}^{n} c_k \varphi_k(x) \right\|^2 = \| f \|^2 - \sum_{k=1}^{n} |c_k|^2,$$

then

$$0 \leqslant \| f \|^2 - \sum_{k=1}^{n} |c_k|^2 < \varepsilon^2,$$

whence it follows that

$$\sum_{k=1}^{\infty} |c_k|^2 = \| f \|^2. \tag{15.3}$$

We have seen earlier (§ 13) that for any normal orthogonal system Bessel's inequality (13.3) holds

$$\sum_{k=1}^{\infty} |c_k|^2 \leqslant \| f \|^2.$$

We now see that in the case of a closed system this inequality changes to the equality (15.3); it is usually known as *Parseval's equality*.

Thus, *if a system is closed, then for any $f(x) \in L^2$ Parseval's equality holds.*

But from this the completeness of the system $\{\varphi_n(x)\}$ in L^2 follows immediately, since if the function $f(x) \in L^2$ is orthogonal to all functions of the system $\{\varphi_n(x)\}$, then

$$c_n = \int_a^b f(x) \bar{\varphi}_n(x) dx = 0 \quad (n = 1, 2, \ldots),$$

i.e. all its Fourier coefficients equal zero; but then $\|f\|^2 = 0$ due to (15.3), i.e.

$$\int_a^b |f|^2 dx = 0,$$

and this is possible only if $f(x) = 0$ almost everywhere.

Thus, the closure of a system in L^2 implies its completeness in L^2; and the proof is concluded.

§ 16. The Riesz–Fischer theorem

We have seen in § 13 that for any function $f(x) \in L^2$ the series $\sum_{n=1}^{\infty} |c_n|^2$, comprising the squares of the moduli of its Fourier coefficients, converges for any orthogonal system. Moreover, in the case when the system under consideration, is complete, then (see 15.3)

$$\|f\|^2 = \sum_{n=1}^{\infty} |c_n|^2.$$

But the following considerably deeper theorem also holds:

THE RIESZ–FISCHER THEOREM. *Let c_n $(n = 1, 2, \ldots)$ be any sequence of numbers for which $\sum_{n=1}^{\infty} |c_n|^2 < +\infty$ and $\{\varphi_n(x)\}$ be any normal orthogonal system. Then there exists an $f(x) \in L^2$ such that the numbers c_n are its Fourier coefficients with respect to this system; if the system is complete, then there exists only one such $f(x)$.*

To prove this we note that if a series $\sum_{n=1}^{\infty} c_n \varphi_n(x)$ is set up, then it should converge in the metric L^2; indeed since $\sum_{n=1}^{\infty} |c_n|^2 < +\infty$, then for any $\varepsilon > 0$, N can be chosen sufficiently large for $\sum_{N+1}^{\infty} |c_n|^2 < \varepsilon$. But then

$$\|S_{n+p}(x) - S_n(x)\|^2 = \sum_{n+1}^{n+p} |c_k|^2 < \varepsilon \quad (n \geqslant N,\ p > 0)$$

(we have already carried out a similar argument in § 14); therefore, the sequence $S_n(x)$ converges in the metric L^2. Thus, an $f(x)$ is found such that $\|f(x) - S_n(x)\|_{L_2} \to 0$ as $n \to \infty$. Repeating the argument of § 15 we see that the series $\sum_{n=1}^{\infty} c_n \varphi_n(x)$ is the Fourier series for $f(x)$, whilst if the system is complete, this $f(x)$ is the only one.

§ 17. The Riesz–Fischer theorem and Parseval's equality for a trigonometric system

Both the Riesz-Fischer theorem and Parseval's equality have been proved for normal systems of functions. Therefore they hold for the system

$$\frac{1}{\sqrt{2\pi}}, \quad \frac{\cos x}{\sqrt{\pi}}, \quad \frac{\sin x}{\sqrt{\pi}}, \ldots, \frac{\cos nx}{\sqrt{\pi}}, \quad \frac{\sin nx}{\sqrt{\pi}}, \ldots$$

Therefore, if a_0, a_n, b_n are a sequence of numbers for which

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) < +\infty, \tag{17.1}$$

then it is possible to find $F(x)$ such that

$$\frac{a_0}{\sqrt{2}} = \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} F(x)\,dx; \quad a_n = \int_{-\pi}^{\pi} F(x) \frac{\cos nx}{\sqrt{\pi}}\,dx; \quad b_n = \int_{-\pi}^{\pi} F(x) \frac{\sin nx}{\sqrt{\pi}}\,dx.$$

Hence, supposing $f(x) = \sqrt{\pi}\, F(x)$, we see that

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx; \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx; \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

Thus, if the series (17.1) converges, then there exists $f(x) \in L^2$, such that the series with coefficients a_n, b_n is a Fourier series.

Parseval's equality for a trigonometric system takes the form

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2). \tag{17.2}$$

We will also note that by virtue of the minimum property of partial sums of a Fourier series (see § 13), we can state particularly for the case of a trigonometric series, that of all the trigonometric polynomials of order not higher than n, the best approximation in the metric L^2 for any $f(x) \in L^2$ is given by the nth partial sum of the series $\sigma(f)$.

In Introductory Material, § 24, we denoted by $E_n^{(p)}(f)$ the best approximation of $f(x) \in L^p$ in the metric L^p by trigonometric polynomials of order not higher than n; this means

$$E_n^{(2)}(f) = \left\{\int_{-\pi}^{\pi} |f(x) - s_n(x)|^2 dx\right\}^{\frac{1}{2}}. \tag{17.3}$$

This formula will be useful later.

§ 18. Parseval's equality for the product of two functions

In this section we will only consider functions which assume real values.

We note yet another useful equality, easily derived from Parseval's equality.

If $f(x) \in L^2$ and $g(x) \in L^2$, and the system $\{\varphi_n(x)\}$ is orthogonal, normal and complete in (a, b), whilst c_n are Fourier coefficients for $f(x)$ and d_n are Fourier coefficients for $g(x)$, then we have the formula

$$\int_a^b f(x)g(x)\,dx = \sum_{n=1}^{\infty} c_n d_n. \tag{18.1}$$

Indeed, if $f \in L^2$ and $g \in L^2$, then this is just as true for their sums and applying Parseval's equality to $f(x)$, $g(x)$ and $f(x) + g(x)$, we have

$$\int_a^b f^2(x)\,dx = \sum_{n=1}^{\infty} c_n^2, \quad \int_a^b g^2(x)\,dx = \sum_{n=1}^{\infty} d_n^2, \tag{18.2}$$

$$\int_a^b [f(x) + g(x)]^2 dx = \sum_{n=1}^{\infty} (c_n + d_n)^2. \tag{18.3}$$

Removing the brackets on the left-hand side of (18.3) we obtain

$$\int_a^b [f(x) + g(x)]^2 dx = \int_a^b f^2(x) dx + 2 \int_a^b f(x)g(x)dx + \int_a^b g^2(x)dx$$

$$= \sum_{n=1}^{\infty} c_n^2 + 2 \sum_{n=1}^{\infty} c_n d_n + \sum_{n=1}^{\infty} d_n^2.$$

Subtracting equation (18.2) from (18.3) and dividing by 2, we obtain the desired formula (18.1).

For the case of the trigonometric system, formula (18.1) takes the form

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)dx = \frac{a_0 \alpha_0}{2} + \sum_{n=1}^{\infty} (a_n \alpha_n + b_n \beta_n),$$

where a_n, b_n are the coefficients for $f(x)$ and α_n, β_n are the coefficients for $g(x)$.

§ 19. The tending to zero of Fourier coefficients

We have seen that if $f(x) \in L^2$, then $\sum_{n=1}^{\infty} |c_n|^2 < +\infty$, whence it immediately follows that $|c_n| \to 0$ as $n \to \infty$. This holds for any orthogonal system. Moreover, the Riesz–Fischer theorem proves that if for some c_n we have $\sum c_n^2 < +\infty$, then these c_n are certainly Fourier coefficients of some function $f(x) \in L^2$.

Matters become considerably more complicated if $f(x) \in L$ but $f^2(x)$ is non-summable. Then we can say very little about the Fourier coefficients of $f(x)$. It would be true to say that given a sequence of numbers c_n for which $\sum c_n^2 = +\infty$, then we do not even know whether there exists a function that possesses these numbers for its Fourier coefficients.

We will state here a few simple facts which will permit us to judge Fourier coefficients to a certain extent.

Mercer's Theorem. *If for an orthogonal normal system*† $\{\varphi_n(x)\}$ *the functions are all bounded, i.e.*

$$|\varphi_n(x)| \leqslant M \quad a \leqslant x \leqslant b \quad (n = 1, 2, \ldots),$$

then the Fourier coefficients of any summable function with respect this system tend to zero.

Let $f(x)$ be summable and $\varepsilon > 0$ be given; we will first find a function $F(x)$, for which $\int_a^b |f(x) - F(x)|\, dx < \varepsilon$, whilst $F(x)$ is bounded. This is always possible from the very definition of a Lebesgue integral.

† Here we are concerned only with functions which assume real values.

Since any bounded function is known to belong to L^2, then its Fourier coefficients tend to zero, which means that for a sufficiently large N we will have

$$\left|\int_a^b F(x)\varphi_n(x)dx\right| < \varepsilon \quad \text{at} \quad n > N.$$

Moreover

$$\left|\int_a^b [f(x) - F(x)]\varphi_n(x)dx\right| \leqslant M\varepsilon.$$

and then

$$\left|\int_a^b f(x)\varphi_n(x)dx\right| < \varepsilon(1 + M) \quad \text{for} \quad n > N,$$

and therefore

$$\int_a^b f(x)\varphi_n(x)dx \to 0 \quad \text{as} \quad n \to \infty,$$

and the theorem is proved.

Since a trigonometric system consists of functions which are all bounded, it follows in particular that

THEOREM. *For any summable function its Fourier coefficients with respect to the trigonometric system tend to zero.*

This fact has very great significance since later (see § 62) we will see that the trigonometric series, the coefficients of which do not tend to zero, can converge only in a set of measure zero. However, the tending to zero of the coefficients of a trigonometric series alone is not sufficient for it to converge (see § 63); moreover, we will later see (Chapter V, § 20) that a Fourier series can also diverge at every point. Thus, the problem of convergence of trigonometric series requires serious investigation.

§ 20. Fejér's lemma

The theorem of § 19 on the tending to zero of the Fourier coefficients is a particular case of the following general result, due to Fejér[1].

FEJÉR'S LEMMA.† *If $f(x) \in L$ has a period 2π and $g(x)$ has a period 2π and is bounded then*

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} f(x)g(nx)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \cdot \int_{-\pi}^{\pi} g(x)dx. \tag{20.1}$$

Here $n \to \infty$, assuming any values, not only integers (supposing $g(x) = \cos x$ or $g(x) = \sin x$, we immediately see that the assertion concerning Fourier coefficients is true).

† This lemma can be omitted on a first reading. It is only used in Chapter XIII.

For the proof of Fejér's lemma, let us note first that if for any $\varepsilon > 0$ it is possible to find $\varphi(x)$ such that

$$\int_{-\pi}^{\pi} |f(x) - \varphi(x)|\, dx < \varepsilon \tag{20.2}$$

and if for $\varphi(x)$ equality (20.1) has already been proved, then it is also true for $f(x)$.

In fact, we have for any n

$$\left| \int_{-\pi}^{\pi} f(x) g(nx)dx - \int_{-\pi}^{\pi} \varphi(x) g(nx) dx \right| < M\varepsilon, \tag{20.3}$$

where M is the upper bound of $f(x)$ in $[-\pi, \pi]$. Moreover, if (20.1) is true for $\varphi(x)$, then N is found such that

$$\left| \int_{-\pi}^{\pi} \varphi(x) g(nx) dx - \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(x) dx \int_{-\pi}^{\pi} g(x) dx \right| < \varepsilon \quad \text{for} \quad n > N. \tag{20.4}$$

Finally, it follows from (20.2) that

$$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \int_{-\pi}^{\pi} g(x)\, dx - \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(x) dx \int_{-\pi}^{\pi} g(x) dx \right|$$

$$< \frac{\varepsilon}{2\pi} \left| \int_{-\pi}^{\pi} g(x) dx \right| < \varepsilon M. \tag{20.5}$$

Therefore, from (20.3), (20.4) and (20.5)

$$\left| \int_{-\pi}^{\pi} f(x) g(nx) dx - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \int_{-\pi}^{\pi} g(x) dx \right| < (2M + 1)\varepsilon \tag{20.6}$$

for any $n > N$. Since ε can be as small as desired, (20.1) follows from (20.6).

Since the class of step-functions is everywhere dense in the class of functions $f \in L$ (see Introductory Material, § 21), then on the basis only of what has already been proved we see that it is sufficient to prove equality (20.1) for step-functions. But this is also easily proved for them, since the interval $[-\pi, \pi]$ is divided into a finite number of intervals in each of which $f(x)$ is constant; then if δ_j is such an interval $f(x) = c_j$ in it and k is the number of intervals δ_j, equality (20.1) takes the form

$$\lim_{n \to \infty} \sum_{j=1}^{k} c_j \int_{\delta_j} g(nx) dx = \sum_{j=1}^{k} c_j \delta_j \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x) dx \tag{20.7}$$

it will be proved if we will satisfy ourselves that for any interval δ

$$\lim_{n \to \infty} \int_{\delta} g(nx) dx = \frac{\delta}{2\pi} \int_{-\pi}^{\pi} g(x) dx. \tag{20.8}$$

Let $\delta = (a, b)$. We have $-\pi \leqslant a < b \leqslant \pi$. It must be proved that

$$\lim_{n\to\infty} \frac{1}{b-a} \int_a^b g(nx)dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x)dx, \tag{20.9}$$

taking into account that $|g(x)| < M$ and $g(x)$ is periodic with period 2π.

For this purpose we will note first of all that

$$\frac{1}{b-a} \int_a^b g(nx)dx = \frac{1}{n(b-a)} \int_{na}^{nb} g(t)dt. \tag{20.10}$$

Let m_1 and m_2 be integers (each of them can be positive, negative or equal to zero) such that

$$\left.\begin{aligned} m_1 \cdot 2\pi \leqslant na < (m_1 + 1)\, 2\pi, \\ m_2 \cdot 2\pi \leqslant nb < (m_2 + 1)\, 2\pi. \end{aligned}\right\} \tag{20.11}$$

Since

$$\int_{na}^{nb} g(t)dt = \int_{m_1\cdot 2\pi}^{m_2\cdot 2\pi} g(t)dt + \int_{m_2\cdot 2\pi}^{nb} g(t)dt - \int_{m_1\cdot 2\pi}^{na} g(t)dt \tag{20.12}$$

and the range of integration of the last two integrals for formula (20.12) does not exceed 2π, then

$$\left| \int_{na}^{nb} g(t)dt - \int_{2\pi m_1}^{2\pi m_2} g(t)dt \right| < 4M\pi, \tag{20.13}$$

Moreover

$$\int_{2\pi m_1}^{2\pi m_2} g(t)dt = (m_2 - m_1) \int_{-\pi}^{\pi} g(t)dt \tag{20.14}$$

because of the periodicity of $g(t)$. From (20.13) and (20.14) this means that

$$\left| \int_{na}^{nb} g(t)dt - (m_2 - m_1) \int_{-\pi}^{\pi} g(t)dt \right| < 4M\pi. \tag{20.15}$$

But from (20.11)

$$(m_2 - m_1 - 1)\, 2\pi < n(b-a) < (m_2 - m_1 + 1)\, 2\pi,$$

and therefore

$$n(b-a) = (m_2 - m_1 + \theta)\, 2\pi, \quad \text{where} \quad |\theta| < 1.$$

In other words,

$$m_2 - m_1 = \frac{n(b-a)}{2\pi} - \theta, \tag{20.16}$$

and therefore from (20.15) and (20.16)

$$\left| \frac{1}{n(b-a)} \int_{na}^{nb} g(t)dt - \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t)dt + \frac{\theta}{n(b-a)} \int_{-\pi}^{\pi} g(t)dt \right| < \frac{4M\pi}{n(b-a)}.$$

Hence it follows that

$$\lim_{n\to\infty}\frac{1}{n(b-a)}\int_{nb}^{na} g(t)dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)dt$$

and taking into account (20.10) we see that (20.9) is proved and thus the proof of the lemma is concluded.

§ 21. Estimate of Fourier coefficients in terms of the integral modulus of continuity of the function

We have seen in § 19 that for any summable function $f(x)$ the Fourier coefficients a_n, b_n tend to zero as $n\to\infty$. However, sometimes the knowledge of this one fact is insufficient and the rate at which it tends to zero should be estimated.

Let us recall that in Introductory Material, § 25 we defined the concept of the integral modulus of continuity $\omega_1(\delta, f)$ for $f(x)$ and we proved that for any $f\in L$ we have $\omega_1(\delta, f)\to 0$ as $\delta\to 0$.

Let c_n be complex Fourier coefficients of the function $f(x)$, i.e.

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx \quad (n = 0, \pm 1, \pm 2, \ldots). \tag{21.1}$$

By substituting $x + \pi/n$ for x we can write

$$c_n = -\frac{1}{2\pi}\int_0^{2\pi} f\left(x+\frac{\pi}{n}\right)e^{-inx}dx. \tag{21.2}$$

Adding (21.2) and (21.2) and dividing by two, we obtain

$$c_n = \frac{1}{4\pi}\int_0^{2\pi}\left[f(x) - f\left(x+\frac{\pi}{n}\right)\right]e^{-inx}dx,$$

whence

$$|c_n| \leqslant \frac{1}{4\pi}\int_0^{2\pi}\left|f\left(x+\frac{\pi}{n}\right) - f(x)\right|dx \leqslant \frac{1}{4\pi}\omega_1\left(\frac{\pi}{n}, f\right).$$

Thus for complex Fourier coefficients of the function $f(x)$ we have

$$|c_n| \leqslant \frac{1}{4\pi}\omega_1\left(\frac{\pi}{n}, f\right) \quad (n = \pm 1, \pm 2, \ldots). \tag{21.3}$$

In the case of real Fourier coefficients, arguing in just the same way we have

$$\left.\begin{aligned} |a_n| &\leqslant \frac{1}{2\pi}\omega_1\left(\frac{\pi}{n}, f\right), \\ & \qquad\qquad\qquad (n = 1, 2, \ldots). \\ |b_n| &\leqslant \frac{1}{2\pi}\omega_1\left(\frac{\pi}{n}, f\right), \end{aligned}\right\} \tag{21.4}$$

Formulae (21.4) give new evidence of the fact that the Fourier coefficients of any $f(x) \in L$ tend to zero but they also permit the rate of this tendency to be judged in terms of the properties of the function, since, roughly speaking, the "better" the function is, the more rapidly its integral modulus of continuity tends to zero.

If $f(x)$ is periodic and continuous in $[-\pi, \pi]$, then from the definition of the modulus of continuity (see Introductory Material, § 25) we immediately conclude that

$$\omega_1(\delta, f) \leqslant \omega(\delta, f) \cdot 2\pi,$$

and therefore for continuous $f(x)$ we have

$$\left.\begin{aligned} |a_n| &\leqslant \omega\left(\frac{\pi}{n}, f\right), \\ & \qquad\qquad (n = 1, 2, \ldots). \\ |b_n| &\leqslant \omega\left(\frac{\pi}{n}, f\right), \end{aligned}\right\} \tag{21.5}$$

§ 22. Fourier coefficients for functions of bounded variation

Let $f(x)$ be a function of bounded variation in $[0, 2\pi]$. If V is its complete variation in $[0, 2\pi]$, then we have

$$\sum_{k=1}^{2n} \left| f\left(x + k\frac{\pi}{n}\right) - f\left(x + (k-1)\frac{\pi}{n}\right) \right| \leqslant V. \tag{22.1}$$

But by arguing as in § 21 we have

$$|a_n| \leqslant \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(x + \frac{n}{\pi}\right) - f(x) \right| dx,$$

$$|b_n| \leqslant \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(x + \frac{\pi}{n}\right) - f(x) \right| dx,$$

and since, because of the periodicity of $f(x)$, we have for any k

$$\int_0^{2\pi} \left| f\left(x + k\frac{\pi}{n}\right) - f\left(x + (k-1)\frac{\pi}{n}\right) \right| dx = \int_0^{2\pi} \left| f\left(x + \frac{\pi}{n}\right) - f(x) \right| dx,$$

then it is also possible to write

$$|a_n| \leqslant \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(x + k\frac{\pi}{n}\right) - f\left(x + (k-1)\frac{\pi}{n}\right) \right| dx.$$

Adding all such inequalities for $k = 1, 2, \ldots, 2n$ and then dividing by $2n$ and taking into account (22.1) we find

$$|a_n| \leqslant \frac{1}{4\pi n} \cdot \int_0^{2\pi} V\, dx = \frac{V}{2n} \tag{22.2}$$

and similarly

$$|b_n| \leqslant \frac{V}{2n}. \tag{22.3}$$

Hence we conclude: *for any function of bounded variation*

$$a_n = O\left(\frac{1}{n}\right), \quad b_n = O\left(\frac{1}{n}\right) \tag{22.4}$$

(for the notation $O(1/n)$ see Introductory Material, § 11).

If it is required that apart from being of bounded variation $f(x)$ is also continuous, then the question arises whether it is not possible to better this estimate? We will prove that this is not so in Chapter II, § 2).

§ 23. Formal operations on Fourier series

We have seen (see § 11) that the trigonometric system is complete in L, i.e. two summable functions can possess identical Fourier series only if they are equal almost everywhere. Thus a Fourier series, even if it is not convergent, is nevertheless closely connected with only one function. We will now demonstrate that it is possible to carry out just the same operations on divergent Fourier series as on series convergent to those functions of which they are the Fourier series.

(1) *Addition and subtraction of Fourier series.* If we have to construct the Fourier series of the sum or difference of two functions, then it is sufficient to add (or subtract) the Fourier series of these functions. Indeed, if

$$f(x) \sim \sum_{n=-\infty}^{n=+\infty} c_n e^{inx}$$

and

$$g(x) \sim \sum_{n=-\infty}^{n=+\infty} \gamma_n e^{inx},$$

then

$$f(x) \pm g(x) \sim \sum_{n=-\infty}^{n=+\infty} (c_n \pm \gamma_n) e^{inx},$$

since

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} [f(x) \pm g(x)] e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx \pm \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x) e^{-inx}\,dx = c_n \pm \gamma_n.$$

Thus if the Fourier series were written in a real form, we should be satisfied that if a_n and b_n are the Fourier coefficients for $f(x)$ and c_n and d_n are the Fourier coefficients for $g(x)$, then for $f(x) \pm g(x)$ the coefficients have the form $a_n \pm c_n$ and $b_n \pm d_n$.

(2) *Multiplication by a constant.* It is immediately evident that if

$$f(x) \sim \sum_{n=-\infty}^{n=+\infty} c_n e^{inx},$$

then

$$kf(x) \sim \sum_{n=-\infty}^{n=+\infty} k c_n e^{inx},$$

where k is any constant. The proof is carried out just as in the preceding case.

(3) *Fourier series for* $f(x+\alpha)$. If α is any constant, then from

$$f(x) \sim \sum c_n e^{inx}$$

it follows that

$$f(x+\alpha) \sim \sum (c_n e^{in\alpha}) e^{inx} \sim \sum c_n e^{in(x+\alpha)}.$$

Indeed

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+\alpha)e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-in(t-\alpha)}dt = e^{in\alpha}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt.$$

Therefore the Fourier series for $f(x+\alpha)$ has just the same form as if we substituted $x+\alpha$ for x in the Fourier series for $f(x)$.

The reader can easily satisfy himself that this is the result if the Fourier series is given in a real form.

(4) *Fourier series for* $f(x)\, e^{imx}$ *where* m *is an integer.* We have

$$f(x)e^{imx} \sim \sum_{n=-\infty}^{n=+\infty} c_{n-m} e^{inx},$$

since

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{imx}e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-i(n-m)x}dx.$$

Hence it again follows that the Fourier coefficients are determined just as if we had operated directly with the series as with a convergent one; in this case we would have

$$f(x)e^{imx} = \sum_{n=-\infty}^{n=+\infty} c_n e^{inx}e^{imx} = \sum_{n=-\infty}^{n=+\infty} c_n e^{i(n+m)x} = \sum_{k=-\infty}^{k=+\infty} c_{k-m} e^{ikx}$$

(5) *Fourier series for* $\bar{f}(x)$. If

$$f(x) \sim \sum c_n e^{inx},$$

then

$$\bar{f}(x) = \sum \bar{c}_n e^{inx},$$

which is verified directly from the Fourier formulae.

(6) *Fourier series for a "convolution"*. It is given that $f(x)$ and $g(x)$ are two periodic functions,

$$f(x) \in L[-\pi, \pi] \quad \text{and} \quad g(x) \in L[-\pi, \pi].$$

Let us consider the product $f(x + t)\, g(t)$. If no additional limitations are put on $f(x)$ and $g(x)$, then it might appear to be a non-summable function of the variable t. But we will prove, following Young[3], that this product for almost all x is a summable function of t in $[-\pi, \pi]$, and supposing

$$Q(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x + t) g(t) dt, \tag{23.1}$$

we have $Q(x) \in L\,[0, 2\pi]$. This function $Q(x)$ is known as the *convolution* for $f(x)$ and $g(x)$.

It is sufficient to consider the case when $f(x) \geqslant 0$ and $g(x) \geqslant 0$.

We assume

$$F(x) = \int_{-\pi}^{x} f(t) dt.$$

Then the function

$$\int_{-\pi}^{\pi} [F(x + t) - F(t - \pi)] g(t) dt = \int_{-\pi}^{\pi} dt \left[\int_{-\pi}^{x} f(t + u) g(t) du \right]$$

exists and is finite for any x. We will suppose that

$$f(t, u, M) = \begin{cases} f(t + u) g(t), & \text{if } f(t + u) g(t) \leqslant M, \\ M, & \text{if } f(t + u) g(t) > M. \end{cases}$$

We have

$$\int_{-\pi}^{\pi} dt \int_{-\pi}^{x} f(t + u) g(t) du = \int_{-\pi}^{\pi} dt \lim_{M \to \infty} \int_{-\pi}^{x} f(t, u, M) du$$

$$= \lim_{M \to \infty} \int_{-\pi}^{\pi} dt \int_{-\pi}^{x} f(t, u, M)\, du = \lim_{M \to \infty} \int_{-\pi}^{x} du \int_{-\pi}^{\pi} f(t, u, M)\, dt$$

$$= \int_{-\pi}^{x} du \lim_{M \to \infty} \int_{-\pi}^{\pi} f(t, u, M)\, dt. \tag{23.2}$$

The limit $\lim\limits_{M \to \infty} \int_{-\pi}^{\pi} f(t, u, M)\, dt$ might seem to be equal to $+\infty$, but because of the equality (23.2) this can occur only for points of some set of measure zero. At those points, where it is finite, it equals $\int_{-\pi}^{\pi} f(t + u)\, g(t)\, dt$.

Thus we have proved that the convolution $Q(x)$ is almost everywhere defined and summable. Now we will express the coefficients of its Fourier series in terms of the coefficients of the series for $f(x)$ and $g(x)$.

If

$$f(x) \sim \sum c_n e^{inx},$$

$$g(x) \sim \sum d_n e^{inx},$$

then the Fourier coefficients μ_n for $Q(x)$ have the form

$$\mu_n = c_n d_{-n}. \tag{23.3}$$

Indeed

$$\mu_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} Q(x)e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t)g(t)dt\right\}e^{-inx}dx.$$

Changing the order of the integration, we obtain

$$\mu_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t)e^{-inx}dx\right\}dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(z)e^{-in(z-t)}\,dz\right\}dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)e^{int}c_n dt = c_n d_{-n}.$$

(Changing the order of integration is valid here, since according to Fubini's theorem (see Introductory Material, § 18) such a change can always be carried out for non-negative summable functions, but $e^{-inx} = \cos nx - i\sin nx$, and $\cos nx$ and $\sin nx$ change sign only a finite number of times in $[-\pi,\pi]$, therefore the integrals under consideration reduce to those for which the rearrangement of the order is valid.)

Thus

$$Q(x) \sim \sum c_n d_{-n}e^{inx}. \tag{23.4}$$

It is appropriate to note here that if $f(x)\in L^2$ and $g(x)\in L^2$, then $\sum |c_n|^2 < +\infty$ and $\sum |d_n|^2 < +\infty$, and therefore $\sum |c_n d_{-n}| < +\infty$. Now we will show that under the given conditions $Q(x)$ is continuous. For this we first partition $g(t)$ into two terms, $g(t) = g_1(t) + g_2(t)$ so that $g_1(t)$ is bounded and $\int_{-\pi}^{\pi} g_2^2(t)\,dt < \varepsilon^2$, where $\varepsilon > 0$ is given. We have

$$Q(x+h) - Q(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x+t+h) - f(x+t)]g_1(t)dt$$

$$+ \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x+t+h) - f(x+t)]g_2(t)dt = I_1 + I_2.$$

If $|g_1(t)| \leqslant M$ $(0 \leqslant t \leqslant 2\pi)$, then

$$|I_1| \leqslant \frac{M}{2\pi}\int_{-\pi}^{\pi}|f(x+t+h) - f(x+t)|\,dt$$

$$= \frac{M}{2\pi}\int_{-\pi}^{\pi}|f(t+h) - f(t)|\,dt \leqslant \frac{M}{2\pi}\,\omega_1(\delta,f)$$

for $0 \leqslant |h| \leqslant \delta$, where $\omega_1(\delta,f)$ is the integral modulus of continuity of the function $f(x)$ and signifies that I_1 can be made as small as desired, if δ is sufficiently small.

For I_2 we find that

$$|I_2| \leqslant \frac{1}{2\pi}\sqrt{\int_{-\pi}^{\pi}[f(x+t+h)-f(x+t)]^2dt}\sqrt{\int_{-\pi}^{\pi}g_2^2(t)dt} \leqslant \frac{\varepsilon}{\pi}\sqrt{\int_{-\pi}^{\pi}f^2(t)\,dt}.$$

Thus, $|Q(x+h)-Q(x)|$ can be made as small as desired, if h is sufficiently small.

We will now remark that since $Q(x)$ is continuous and the series (23.4) converges absolutely and uniformly, then this series by virtue of the theorem in § 12 converges to $Q(x)$ at every point. In particular, we derive from this, by supposing $x = 0$,

$$Q(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)g(t)dt = \sum_{n=-\infty}^{n=+\infty} c_n d_{-n}. \tag{23.5}$$

(7) *Fourier series for a product.* Let

$$f(x) \sim \sum_{n=-\infty}^{n=+\infty} c_n e^{inx}, \quad g(x) \sim \sum_{n=-\infty}^{n=+\infty} d_n e^{inx}.$$

We assume that $f(x) \in L^2$ and $g(x) \in L^2$. Then $f(x)\,g(x) \in L$.

Supposing

$$f(x)g(x) \sim \sum_{n=-\infty}^{n=+\infty} \gamma_n e^{inx},$$

we will show that

$$\gamma_n = \sum_{k=-\infty}^{k=+\infty} c_k d_{n-k}. \tag{23.5'}$$

In order to succeed in doing this, we note that

$$\gamma_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)g(x)e^{-inx}dx,$$

and therefore, supposing

$$h(x) = g(x)e^{-inx}, \tag{23.6}$$

we have

$$\gamma_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)h(x)dx.$$

If we denote the Fourier coefficients of $h(x)$ by μ_n, then according to formula (23.4)

$$\gamma_n = \sum_{k=-\infty}^{k=+\infty} c_k \mu_{-k}. \tag{23.7}$$

But since on the basis of item (4) of this section it follows from (23.6) that

$$\mu_k = d_{k+n},$$

then

$$\gamma_n = \sum_{k=-\infty}^{k=+\infty} c_k d_{n-k},$$

and this is formula (23.5′) which we wanted to prove.

Note. We will recall that for numerical series the following theorem is valid: if $u_0 + u_1 + \cdots + u_n + \cdots$ absolutely converges and its sum equals u, and $v_0 + v_1 + \cdots + v_n + \cdots$ absolutely converges and its sum equals v, then the series

$$u_0 v_0 + (u_0 v_1 + v_0 u_1) + \cdots + (u_0 v_n + u_1 v_{n-1} + \cdots + u_n v_0) + \cdots$$

absolutely converges and its sum is uv.

It is not difficult to prove that if we established a series for the product $f(x)\, g(x)$ using this formula for the multiplication of the series, then the coefficients of this series would be expressed by the formula (23.7), i.e. we see that Fourier series can be treated here in the same way as if they converged absolutely.

Conclusion. If we have $\sum |c_n| < +\infty$ and $\sum |d_n| < +\infty$, then $\sum |\gamma_n| < +\infty$, since it is known that the product of two absolutely convergent series converge absolutely; moreover, $\sum |\gamma_n| \leqslant \sum |c_n| \sum |d_n|$, since in an absolutely convergent series it is possible to rearrange its terms, without altering its sum.

Later (see § 61) we will see that absolute convergence of a trigonometric series in $[-\pi, \pi]$ occurs when and only when the series of the absolute values of its coefficients converges. Therefore we have

THEOREM. *If $f(x)$ and $g(x)$ expand into absolutely convergent series, then their product also possesses this property.*

(8) *Integration of Fourier series.* Let $f(x)$ be a periodic summable function, and $F(x)$ its indefinite Lebesgue integral

$$F(x) = C + \int_0^x f(t)dt.$$

We set ourselves the task of expanding $F(x)$ into a Fourier series, if the series for $f(x)$ has already been obtained:

$$f(x) \sim \sum_{n=-\infty}^{n=+\infty} c_n e^{inx}.$$

We note above all that

$$F(2\pi) - F(0) = \int_0^{2\pi} f(t)\, dt = 2\pi c_0,$$

and therefore if $c_0 \neq 0$, then $F(x)$ will not be periodic. Therefore, we shall consider the auxiliary function

$$\Phi(x) = F(x) - c_0 x. \tag{23.8}$$

Since

$$\Phi(x + 2\pi) = F(x + 2\pi) - c_0(x + 2\pi) = C + \int_0^{2\pi+x} f(t)\, dt - c_0 x - c_0 2\pi$$

$$= C + \int_0^x f(t)\, dt - c_0 x = \Phi(x),$$

then $\Phi(x)$ is already periodic. It is absolutely continuous as is also $F(x)$ and

$$\Phi'(x) = F'(x) - c_0 = f(x) - c_0$$

almost everywhere.

Let us find the Fourier coefficients for $\Phi(x)$; we have for $n \neq 0$

$$C_n = \frac{1}{2\pi}\int_0^{2\pi} \Phi(x)e^{-inx}dx$$

$$= \frac{1}{2\pi}\left\{\frac{\Phi(x)e^{-inx}}{-in}\right\}\Bigg|_0^{2\pi} + \frac{1}{2\pi in}\int_0^{2\pi} f(x)e^{-inx}\,dx - \frac{c_0}{2\pi in}\int_0^{2\pi} e^{-inx}dx \quad (23.9)$$

(integration by parts is valid because of the absolute continuity of $\Phi(x)$). Since $\Phi(2\pi) = \Phi(0)$, then we obtain immediately

$$C_n = \frac{c_n}{in}, \quad (n = \pm 1, \pm 2, \ldots). \quad (23.10)$$

We can now write

$$\Phi(x) \sim C_0 + \sum{}' \frac{c_n}{in} e^{inx}, \quad (23.11)$$

where the symbol $\sum'$ denotes that the term with $n = 0$ is omitted.

From (23.8) and (23.11) we conclude that

$$F(x) - c_0 x \sim C_0 + \sum{}' \frac{c_n}{in} e^{inx}. \quad (23.12)$$

It is clear that if we were to integrate completely formally the series $\sigma(f)$, then we would obtain this same series (23.12) for $F(x)$.

If the series for $f(x)$ were written in the real form

$$f(x) \sim \frac{a_0}{2} + \sum (a_n \cos nx + b_n \sin nx),$$

then we would obtain

$$F(x) - \frac{a_0}{2}x \sim C + \sum \frac{-b_n \cos nx + a_n \sin nx}{n}.$$

(9) *Differentiation of Fourier series. Fourier–Stieltjes series.* Let $F(x)$ be absolutely continuous in $[0, 2\pi]$ and have a period 2π. If

$$F(x) \sim \sum c_n e^{inx},$$

then for its derivative we have

$$F'(x) \sim \sum in c_n e^{inx}. \quad (23.13)$$

Indeed, it is sufficient to apply formula (23.10), assuming that $f(x) = F'(x)$.

Thus the Fourier series for the derivative of $F(x)$ is obtained in the same way as if we differentiated the Fourier series for $F(x)$.

Similarly, if

$$F(x) \sim \frac{a_0}{2} + \sum (a_n \cos nx + b_n \sin nx),$$

then

$$F'(x) \sim \sum n(b_n \cos nx - a_n \sin nx).$$

We note, however, that these formulae are valid only if $F(x)$ is absolutely continuous, otherwise it is not an indefinite Lebesgue integral of its derivative, even if this derivative exists and is summable.

In the case when $F(x)$ is a function of bounded variation, then, supposing

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-inx} dF \quad (n = 0, \pm 1, \ldots), \tag{23.14}$$

where the integral in formula (23.14) is a Riemann–Stieltjes integral (see Introductory Material, § 16), we write

$$dF \sim \sum c_n e^{inx} \tag{23.15}$$

and this series (23.15) is known as *the Fourier–Stieltjes series for dF.*

If we assume

$$\Phi(x) = F(x) - c_0 x,$$

then $\Phi(x)$ is also of bounded variation and is periodic too. Let C_n be the Fourier coefficients for $\Phi(x)$; then for $n \neq 0$, integrating by parts, we find

$$C_n = \frac{1}{2\pi} \int_0^{2\pi} \Phi(x)\, e^{-inx}\, dx = \frac{1}{2\pi\, in} \int_0^{2\pi} e^{-inx}\, d\Phi = \frac{c_n}{in},$$

since $d\Phi = dF - c_0\, dx$. Therefore, if

$$\Phi(x) \sim C_0 + \sum{}' C_n\, e^{inx},$$

where the symbol $\sum{}'$ indicates that the term at $n = 0$ is omitted, then

$$\Phi(x) \sim C_0 + \sum{}' \frac{c_n}{in}\, e^{inx}$$

and

$$F(x) - C_0 x \sim C_0 + \sum{}' \frac{c_n}{in}\, e^{inx}. \tag{23.16}$$

From formulae (23.15) and (23.16) it follows that *the Fourier–Stieltjes series for dF agrees accurately as regards the constants with the result of differentiating the Fourier series for $F(x) - c_0 x$.*

§ 24. Fourier series for repeatedly differentiated functions

Let us assume that $k \geqslant 2$, the function $f(x)$ has derivatives up to the order $k - 1$ inclusive and the derivative of the $(k - 1)$th order is absolutely continuous; then the kth derivative is summable. Denoting the Fourier coefficients for $f^{(k)}(x)$ by $c_n^{(k)}$, we find from formula (23.10) that

$$c_n^{(k-1)} = \frac{c_n^{(k)}}{in}; \quad c_n^{(k-2)} = \frac{c_n^{(k-1)}}{in} = \frac{c_n^{(k)}}{(in)^2},$$

etc., and finally

$$c_n = \frac{c_n^{(k)}}{(in)^k}.$$

Hence it is immediately clear that the higher the derivative of the function is, then the more rapidly do its Fourier coefficients tend to zero.

In particular, if $f^{(k)}(x)$ is defined and summable almost everywhere, then $c_n^{(k)}$ tends to zero as $n \to \pm\infty$, just as the Fourier coefficients of a summable function and then

$$c_n = o\left(\frac{1}{|n|^k}\right). \tag{24.1}$$

Such an estimate does of course occur if the Fourier series has a real form, i.e.

$$a_n = o\left(\frac{1}{n^k}\right) \quad \text{and} \quad b_n = o\left(\frac{1}{n^k}\right). \tag{24.2}$$

§ 25. On Fourier coefficients for analytic functions

Let $f(x)$ be a function of a real variable, analytic in the interval $[-\pi, \pi]$ and periodic with period 2π. Let us estimate its Fourier coefficients. We will show that they decrease at the rate of a geometric progression; more exactly, it is possible to find θ, $0 < \theta < 1$, and a constant A such that

$$|c_n| \leqslant A\theta^{|n|} \quad (n = 0, \pm 1, \pm 2, \ldots) \tag{25.1}$$

or in a real form

$$|a_n| \leqslant A\theta^n \quad \text{and} \quad |b_n| \leqslant A\theta^n \quad (n = 0, 1, 2, \ldots). \tag{25.2}$$

The numbers θ and A vary, generally speaking, with the function $f(x)$ being considered.

In order to prove this, we note first of all that because of the conditions imposed on $f(x)$, we have

$$f(-\pi) = f(\pi) \quad \text{and} \quad f^{(k)}(-\pi) = f^{(k)}(\pi) \quad (k = 1, 2, \ldots).$$

In estimating the Fourier coefficients for functions possessing k derivatives, we have seen (see § 24) that

$$|c_n| = \frac{1}{|n|^k}|c_n^{(k)}|,$$

where $c_n^{(k)}$ are the Fourier coefficients of $f^{(k)}(x)$. But

$$c_n^{(k)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^{(k)}(x)\, e^{-inx}\, dx.$$

Therefore, if the maximum modulus of $f^{(k)}(x)$ is denoted by M_k, then

$$|c_n| \leqslant \frac{M_k}{|n|^k}.$$

But for the numbers M_k, this inequality holds

$$M_k < B^k k! \quad (k = 1, 2, \ldots),$$

where B is a constant.† Therefore

$$|c_n| \leqslant \frac{B^k k!}{|n|^k} \leqslant \left(\frac{Bk}{|n|}\right)^k \quad (n = \pm 1, \pm 2, \ldots) \tag{25.3}$$

Let us choose p such that

$$\frac{B}{p} < 1, \tag{25.4}$$

and suppose that

$$\theta_1 = \frac{B}{p}. \tag{25.5}$$

The number k in formula (25.3) is at our disposal, since the function $f(x)$ possesses derivatives of all orders. Therefore, for given n and p we can find an integer k from the condition

$$k \leqslant \frac{|n|}{p} < k + 1.$$

If this is so, then $|n| \geqslant pk$ and taking into account (25.3) and (25.5)

$$|c_n| \leqslant \left(\frac{B}{p}\right)^k = \theta_1^k = \frac{\theta_1^{k+1}}{\theta_1} < \frac{\theta_1^{|n|/p}}{\theta_1}, \tag{25.6}$$

whilst by virtue of (25.4) and (25.5) we have $\theta_1 < 1$; denoting by θ the number which satisfies the condition

$$\theta_1^{1/p} < \theta < 1, \tag{25.7}$$

and supposing

$$A = \frac{1}{\theta_1},$$

we have from (25.6) and (25.7)

$$|c_n| < A\theta^{|n|} \quad (n = 0, 1, 2, \ldots),$$

and this is what was required to be proved (see (25.1)).

---

† Indeed, from the assumptions made regarding $f(x)$, it follows that it is possible to expand it analytically in some plane region containing the interval $[-\pi, \pi]$. If we denote by C an arbitrary rectifiable contour enclosing the interval $[-\pi, \pi]$ and lying in the region where $f(z)$ is analytic, then according to Cauchy's formula

$$f^{[k]}(x) = \frac{k!}{2\pi i} \int_C \frac{f(z)}{(z-x)^{k+1}} dz.$$

If the length of the contour C is l, $\max\limits_C |f(z)| = M$ and the minimum distance of the points z on C from the points x in $[-\pi, \pi]$ equals δ, then

$$|f^{(k)}(x)| \leqslant Ml \frac{1}{2\pi} k! \frac{1}{\delta^{k+1}} < B^k k!,$$

if B is chosen so that $B > 1/\delta$ and $B > Ml/8\pi\delta^2$.

If we take a Fourier series in its real form, then the inequality takes the form

$$|a_n| \leqslant A\theta^n \quad \text{and} \quad |b_n| \leqslant A\theta^n \quad (n = 0, 1, 2, \ldots).$$

The reverse statement is also true, namely: if for a function $f(x)$ the Fourier coefficients satisfy the inequality (25.1) where A is a constant and $0 < \theta < 1$, then $f(x)$ is an analytic function in the interval $[-\pi, \pi]$.

Indeed, the series $\sum |c_n| < +\infty$ and we then have

$$f(x) = \sum_{n=-\infty}^{n=+\infty} c_n e^{inx}.$$

Differentiating this equality k times, where k is any number, we obtain

$$f^{(k)}(x) = \sum_{n=-\infty}^{n=+\infty} c_n (i)^k n^k e^{inx}.$$

The differentiation term-by-term is valid, since the series obtained converges absolutely and uniformly because

$$|c_n (i)^k n^k| \leqslant A\theta^{|n|} |n|^k,$$

and since k is a constant, then the convergence of the series $\sum \theta^{(n)} |n^k|$ follows if only from the application of Cauchy's test to it.

Thus, $f(x)$ possesses derivatives of all orders. But, moreover,

$$M_k = \max |f^{(k)}(x)| \leqslant 2A \sum_{n=1}^{\infty} \theta^n n^k.$$

Hence it is possible to deduce the validity of the inequality

$$M_k < B^k k!$$

for some B. Indeed

$$\int_0^\infty \theta^x x^k \, dx = -\frac{k}{\ln\theta} \int_0^\infty \theta^x x^{k-1} \, dx = \frac{k(k-1)}{\ln^2\theta} \int_0^\infty \theta^x x^{k-2} \, dx = \cdots = (-1)^k \frac{k!}{\ln^k \theta},$$

which gives the desired inequality.

Now let x_0 be any point in $[-\pi, \pi]$. Let x be any other point for which

$$|x - x_0| < \frac{1}{B}.$$

Using Taylor's formula with a remainder in a Lagrange form

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{f^{(n)}(x_0 + \theta'(x - x_0))}{n!} (x - x_0)^n,$$

where $0 < \theta' < k$. But

$$\left| \frac{f^{(n)}(x_0 + \theta'(x - x_0))}{n!} (x - x_0)^n \right| \leqslant \frac{B^n n!}{n!} (x - x_0)^n = (B |x - x_0|)^n.$$

Because $|x - x_0| < 1/B$ the right-hand side tends to zero as $n \to \infty$ and this means that

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k,$$

i.e. $f(x)$ expands into a Taylor series in the neighbourhood of the point x_0; but x_0 is any point of $[-\pi, \pi]$, which means that $f(x)$ is an analytic function in $[-\pi, \pi]$.

§ 26. The simplest cases of absolute and uniform convergence of Fourier series

We will start with the following simple observations. Let us consider a trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \tag{26.1}$$

If

$$\sum (|a_n| + |b_n|) < +\infty, \tag{26.2}$$

then it converges absolutely (and uniformly) in $[-\pi, \pi]$.

It is useful to note (we have already referred to this in § 23) that the convergence of series (26.2) is not only sufficient but necessary† for the series (26.1) to converge absolutely in $[-\pi, \pi]$.

It now remains to consider some concrete cases when the Fourier series converges absolutely and uniformly. If this occurs, then this series has as its sum the function $f(x)$ for which it serves as Fourier series (see § 12). In particular, it follows that

If $f(x)$ possesses a summable derivative of the second order, then its Fourier series converges uniformly to $f(x)$.

Indeed (see § 24) in this case

$$a_n = O\left(\frac{1}{n^2}\right), \quad b_n = O\left(\frac{1}{n^2}\right).$$

Later we will see that the requirements imposed on $f(x)$ are too limiting and we can obtain uniform convergence for considerably more general assumptions, but it is expedient to mention this theorem, since even in this form it can be useful.

We will mention here yet another simple but important case, where the absolute and uniform convergence of a Fourier series is readily detected, namely:

THEOREM. *If $F(x)$ is absolutely continuous and its derivative $F'(x) = f(x)$ is a function with an integrable square, then the Fourier series of $F(x)$ converges absolutely and uniformly.*

Indeed, in this case, if the Fourier coefficients for $f(x)$ are denoted by a_n, b_n, then $\sum (a_n^2 + b_n^2) < +\infty$ (see § 13) and according to formula (23.10), denoting the Fourier coefficients for $F(x)$ by A_n and B_n, we have

$$|A_n| = \left|\frac{b_n}{n}\right| \text{ and } |B_n| = \left|\frac{a_n}{n}\right|,$$

† In § 61 it will be shown that for the convergence of (26.2) the absolute convergence of (16.1) is sufficient not in the whole interval $[-\pi, \pi]$, but only in a set of positive measure.

and therefore

$$|A_n| \leqslant \frac{1}{2}|b_n|^2 + \frac{1}{2}\frac{1}{n^2} \quad \text{and} \quad |B_n| \leqslant \frac{1}{2}a_n^2 + \frac{1}{2}\frac{1}{n^2}.$$

Consequently

$$\sum_{n=1}^{\infty}(|A_n| + |B_n|) < +\infty,$$

and the theorem is proved.

In § 3 of Chapter IX, this theorem is generalized, so that instead of the hypothesis $f'(x) \subset L^2$ we consider the case $f'(x) \subset L^p(p > 1)$ and we show that the result still holds. A number of considerably stronger theorems on the absolute convergence of Fourier series will also be given there.

A very particular case of the given theorem is given by the following example; if $F(x)$ is represented by a continuous broken line, then its Fourier series converges absolutely and uniformly.

In fact, in this case $F'(x)$ is a function which possesses a derivative everywhere except for a finite number of points and this derivative $f(x)$ consists of a finite number of steps, and is therefore bounded, and consequently $f^2(x)$ is moreover summable.

§ 27. Weierstrass's theorem on the approximation of a continuous function by trigonometric polynomials

Let $f(x)$ be a continuous function in the interval $[-\pi, \pi]$ and $f(-\pi) = f(\pi)$. If we expand it periodically with period 2π, it will be continuous along the whole axis Ox. We define a function with period 2π as a continuous periodic function when and only when it remains continuous after its periodic expansion; if $f(x)$ is continuous only in a certain interval of length 2π, but at its end points assumes different values and therefore becomes discontinuous if it is expanded periodically (see Fig. 4 on page 50), then we will not call it a continuous periodic function.

After this definition we can express a theorem.

WEIERSTRASS'S THEOREM. *For any continuous periodic function $f(x)$ and for any $\varepsilon > 0$ a trigonometric polynomial $T(x)$ can be found such that*

$$|f(x) - T(x)| < \varepsilon \quad (-\infty < x < +\infty). \tag{27.1}$$

A large number of proofs of this important theorem exist. We will refer here to one of them.

Because of the continuity of $f(x)$ in $[-\pi, \pi]$ it is possible to find a δ such that

$$|f(x') - f(x'')| < \frac{\varepsilon}{2} \quad \text{for} \quad |x' - x''| \leqslant \delta, \tag{27.2}$$

where x' and x'' are any two points in $[-\pi, \pi]$.

Let us divide the interval $[-\pi, \pi]$ into m equal parts, choosing m so that $2\pi/m < \delta$. We will denote by $\psi(x)$ the broken line coinciding with $f(x)$ at the points $k\pi/m$, where $k = 0, \pm 1, \ldots, \pm m$, and will assume that $\psi(x + 2\pi) = \psi(x)$ for any $x(-\infty < x < +\infty)$. From (27.2) it is clear that

$$|f(x) - \psi(x)| < \frac{\varepsilon}{2} \quad \text{for} \quad |x' - x''| \leqslant \delta$$

and because of the periodicity of the two functions this is also true for any x, $-\infty < x < +\infty$.

Since $\psi(x)$ is a broken line, then according to the proof at the end of § 26 its Fourier series converges absolutely to it. Therefore denoting the sum of the first n terms of its Fourier series by $S_n(x)$, it is possible to choose n sufficiently large for

$$|\psi(x) - S_n(x)| < \frac{\varepsilon}{2} \quad \text{for} \quad -\infty < x < +\infty.$$

It is clear that $S_n(x)$ is a trigonometric polynomial and denoting it by $T(x)$ we see that the theorem is proved.

§ 28. The density of a class of trigonometric polynomials in the spaces L^p $(p \geqslant 1)$

Weierstrass's theorem which has just been proved can be considered as evidence of the fact that the class of trigonometric polynomials is everywhere dense in the space C of continuous periodic functions.

It follows from this that this class is everywhere dense in any space $L^p (p \geqslant 1)$.

Indeed, if $f(x) \in L^p$, then for any ε (see Introductory Material, § 21) it is possible to find a continuous $\varphi(x)$ such that

$$\|f - \varphi\|_{L^p} \leqslant \varepsilon,$$

and on the other hand it is possible to find a trigonometric polynomial $T(x)$ such that

$$|\varphi(x) - T(x)| < \frac{\varepsilon}{2\pi}, \quad 0 \leqslant x \leqslant 2\pi,$$

and therefore

$$\|\varphi - T\|_{L^p} < \varepsilon$$

(it is assumed that the norm is calculated in an interval of length 2π). Therefore, according to Minkowski's inequality (see Introductory Material, § 10)

$$\|f - T\|_{L^p} < 2\varepsilon,$$

and the theorem is proved.

§ 29. Dirichlet's kernel and its conjugate kernel

An important role in the study of the convergence of trigonometric series is played by the functions

$$D_n(x) = \frac{1}{2} + \cos x + \cdots + \cos nx \tag{29.1}$$

and

$$\bar{D}_n(x) = \sin x + \cdots + \sin nx. \tag{29.2}$$

The function $D_n(x)$ can be written thus:

$$D_n(x) = \frac{\sin\left(n + \frac{1}{2}\right) x}{2 \sin \frac{x}{2}}. \tag{29.3}$$

Indeed

$$\begin{aligned} 2 \sin \frac{x}{2}\, D_n(x) &= \sin \frac{x}{2} + \sum_{k=1}^{n} 2 \sin \frac{x}{2} \cos kx \\ &= \sin \frac{x}{2} + \sum_{k=1}^{n} \left[\sin\left(k + \frac{1}{2}\right) x - \sin\left(k - \frac{1}{2}\right) x\right] \\ &= \sin\left(n + \frac{1}{2}\right) x, \end{aligned}$$

whence after dividing by $2 \sin(x/2)$, formula (29.3) is obtained.

Expression (29.3) is called *the Dirichlet kernel*, since Dirichlet first used it in the study of the convergence of Fourier series (see § 31).

Similarly $\bar{D}_n(x)$ is called *the kernel conjugate to the Dirichlet kernel*; it takes the form

$$\bar{D}_n(x) = \frac{\cos \frac{x}{2} - \cos\left(n + \frac{1}{2}\right) x}{2 \sin \frac{x}{2}}, \tag{29.4}$$

which can also be easily verified directly.

From formulae (29.3) and (29.4) it is immediately evident that if $x \not\equiv 0 \pmod{2\pi}$, then

$$|D_n(x)| \leqslant \frac{1}{2\left[\sin \frac{x}{2}\right]} \tag{29.5}$$

and

$$|\bar{D}_n(x)| \leqslant \frac{1}{\left[\sin \frac{x}{2}\right]}. \tag{29.6}$$

We now note that the function $(\sin x)/x$ decreases in the interval $(0, \pi/2)$ (which it is possible to prove by simple differentiation) and therefore

$$\frac{\sin x}{x} \geqslant \frac{\sin \frac{\pi}{2}}{\left(\frac{\pi}{2}\right)} = \frac{2}{\pi}$$

This means

$$\frac{\sin x}{x} \geqslant \frac{2}{\pi} \quad \text{for} \quad 0 \leqslant x \leqslant \frac{\pi}{2}. \tag{29.7}$$

Using (29.5) and (29.6) we obtain

$$|D_n(x)| \leqslant \frac{\pi}{2x} \quad \text{for} \quad 0 < |x| \leqslant \pi \tag{29.8}$$

and

$$|\bar{D}_n(x)| \leqslant \frac{\pi}{x} \quad \text{for} \quad 0 < |x| \leqslant \pi. \tag{29.9}$$

We will use these formulae frequently later. Most often it will be sufficient to estimate

$$D_n(x) = O\left(\frac{1}{x}\right) \quad \text{and} \quad \bar{D}_n(x) = O\left(\frac{1}{x}\right) \text{ as } x \to 0; \tag{29.10}$$

sometimes it will be important that if $\delta \leqslant |x| \leqslant \pi$, then

$$|D_n(x)| \leqslant \frac{\pi}{2\delta} \quad \text{and} \quad |\bar{D}_n(x)| \leqslant \frac{\pi}{\delta}. \tag{29.11}$$

Because of the periodicity of $D_n(x)$ and $\bar{D}_n(x)$ it is also possible to say that (29.11) holds if $\delta \leqslant x \leqslant 2\pi - \delta$.

§ 30. Sine or cosine series with monotonically decreasing coefficients

Before turning to a study of the cases when the problem of convergence of the trigonometric series requires close examination, we will consider some cases where it is very easy to judge the convergence.

Let us begin with series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{30.1}$$

and

$$\sum_{n=1}^{\infty} b_n \sin nx, \tag{30.2}$$

i.e., series consisting of either cosines only or sines only. We will consider firstly the important case when these series possess monotonically decreasing coefficients tending to zero, which can be denoted thus:

$$a_n \downarrow 0 \quad \text{and} \quad b_n \downarrow 0.$$

In studying these series we will use the estimates of $D_n(x)$ and $\bar{D}_n(x)$ given in § 29 and Abel's lemma (see Introductory Material, § 1). This permits us to prove the theorem.

THEOREM 1. *If $a_n \downarrow 0$, then the series*

$$\frac{a_0}{2} + \sum a_n \cos nx$$

converges everywhere apart, perhaps, from the points $x \equiv 0 \ (mod\ 2\pi)$; at any $\delta > 0$ it converges uniformly in $\delta \leqslant x \leqslant 2\pi - \delta$.

If $b_n \downarrow 0$, then the series

$$\sum b_n \sin nx$$

converges everywhere; at any $\delta > 0$ it converges uniformly in $\delta \leqslant x \leqslant 2\pi - \delta$.

Indeed, supposing in Abel's lemma that

$$u_n = a_n, \quad v_0 = \frac{1}{2} \quad \text{and} \quad v_n(x) = \cos nx \quad (n = 1, 2, \ldots),$$

we have

$$V_n(x) = D_n(x),$$

and since the uniform boundedness of the functions $D_n(x)$ in $\delta \leqslant x \leqslant 2\pi - \delta$ follows from formula (29.11), then the series converges uniformly in this interval. If $0 < x < 2\pi$, then it is always possible to take δ so small that $\delta \leqslant x \leqslant 2\pi - \delta$, which indicates that the series (30.1) converges at the point x.

At $x = 0$, the series (30.1) converges when and only when $\sum a_n < +\infty$.

For the series (30.2), the proof is similar; it is only necessary to substitute $u_n = b_n$ and $v_n(x) = \sin nx$ in Abel's lemma; then $V_n(x) = \overline{D}_n(x)$ and again the application of the inequality (29.11) gives evidence of the uniform convergence of series (30.2) in $\delta \leqslant x \leqslant 2\pi - \delta$, and therefore its convergence at every point, apart from the points $x \equiv 0 \pmod{2\pi}$. But at the latter it also converges because all the terms of the series equal zero.

The theorem is completely proved.

Note. By the generalization of Abel's Lemma (see Introductory Material, § 1) the series (30.1) and (30.2) converge uniformly in $\delta \leqslant x \leqslant 2\pi - \delta$ (this indicates that it also converges in $0 < x < 2\pi$) and when instead of $a_n \downarrow 0$ or $b_n \downarrow 0$ we assume only that $\{a_n\}$ or $\{b_n\}$ is a sequence of bounded variation, $a_n \to 0$ and $b_n \to 0$, moreover.

Let us return to the case of the monotonically decreasing coefficients. It is clear that if

$$a_n \downarrow 0 \quad \text{and} \quad \sum a_n < +\infty,$$

then the series $a_0/2 + \sum a_n \cos nx$ converges absolutely and uniformly in the whole interval $0 \leqslant x \leqslant 2\pi$ (and even for $-\infty < x < +\infty$). On the other hand if the condition $\sum a_n < +\infty$ is not fulfilled, then not only uniform but also simple convergence along the whole axis is not possible, since at the points $x \equiv 0 \pmod{2\pi}$ the series (30.1) diverges.

The question of the uniform convergence of the series $\sum b_n \sin nx$ is decided in another manner. Here we have.

THEOREM 2. *If $b_n \downarrow 0$, then for uniform convergence of the series $\sum b_n \sin nx$ in $[0, 2\pi]$ it is necessary and sufficient that $nb_n \to 0$.*

Necessity condition. If the series (30.2) converges uniformly in $[0, 2\pi]$, then for any $\varepsilon > 0$ it is possible to find m such that

$$\left| \sum_{m+1}^{2m} b_n \sin nx \right| < \varepsilon, \quad 0 \leqslant x \leqslant 2\pi.$$

We will let $x = \pi/4m$; then for $(m + 1) \leqslant n \leqslant 2m$ we have $\pi/4 \leqslant nx \leqslant \pi/2$ and therefore $\sin nx \geqslant \sin \pi/4 = 1/\sqrt{2}$. Consequently,

$$\frac{1}{\sqrt{2}} \sum_{m+1}^{2m} b_n < \varepsilon,$$

and since b_n decrease monotonically, $(1/\sqrt{2})mb_{2m} < \varepsilon$, i.e., $mb_{2m} < \sqrt{2}\varepsilon$, which means that $m\,b_m \to 0$ as $m \to \infty$. The necessity is thus proved.

Sufficiency condition. We already know that series (30.2) converges uniformly in $\delta \leqslant x \leqslant 2\pi - \delta$ for any δ (for the single condition $b_n \downarrow 0$). This means that if we prove that the addition of the condition $nb_n \to 0$ implies uniform convergence in $(-a, a)$, where a is any number > 0, then everything will be proved. Moreover, because of the oddness of $\sin nx$ it is sufficient to take $0 \leqslant x \leqslant a$. We will prove uniform convergence of the series in $0 \leqslant x \leqslant \pi/4$.

Let $\varepsilon_n = \max\limits_{k \geqslant n} kb_k$. It is known that series (30.2) converges for any x; let us define

$$r_n(x) = \sum_{k=n}^{\infty} b_k \sin kx.$$

We will prove that $|r_n(x)| \leqslant K\varepsilon_n$ in $0 \leqslant x \leqslant \pi/4$, where K is a constant whence the uniform convergence of the series (30.2) in $[0, 2\pi]$ follows.

Above all, $r_n(0) = 0$, if $x \neq 0$, then it is always possible to find an integer N such that $1/N < x \leqslant 1/(N-1)$. If $N > n$, then we write

$$r_n(x) = \sum_{k=n}^{N-1} b_k \sin kx + \sum_{k=N}^{\infty} b_k \sin kx = r_n^{(1)}(x) + r_n^{(2)}(x).$$

If $N \leqslant n$, then let $r_n^{(1)}(x) = 0$ and $r_n^{(2)}(x) = r_n(x)$. Let us estimate $r_n^{(1)}(x)$ and $r_n^{(2)}(x)$ separately.

We have, since $|\sin kx| \leqslant k\,|x|$,

$$|r_n^{(1)}(x)| \leqslant \sum_{k=n}^{N-1} kb_k x \leqslant x\varepsilon_n(N-n) \leqslant \frac{N-n}{N-1}\varepsilon_n \leqslant \varepsilon_n.$$

In order to estimate $r_n^{(2)}(x)$ we consider two separate cases:

(1) If $n < N$, then using Abel's transformation (see Introductory Material, § 1) we find

$$|r_n^{(2)}(x)| \leqslant \sum_{k=N}^{\infty} (b_k - b_{k+1})\,|\bar{D}_k(x)| + b_N\,|\bar{D}_{N-1}(x)|.$$

But since (see (29.9))

$$|\bar{D}_k(x)| \leqslant \frac{\pi}{x} \quad \text{for} \quad 0 < |x| \leqslant \pi,$$

then

$$|r_n^{(2)}(x)| \leqslant \frac{2\pi}{x} b_N \leqslant 2\pi N b_N \leqslant 2\pi\varepsilon_n$$

because $n < N$ and because of the definition of ε_n.

(2) If $N \leqslant n$, then $r_n^{(2)}(x) = r_n(x)$ and then calculation shows that

$$|r_n(x)| = |r_n^{(2)}(x)| \leqslant 2\pi\varepsilon_n.$$

Therefore

$$|r_n(x)| \leqslant |r_n^{(1)}(x)| + |r_n^{(2)}(x)| \leqslant (2\pi + 1)\,\varepsilon_n,$$

which means that the desired inequality has been proved.

Note. From the theorem just proved the following conclusion can be deduced immediately:

There exist trigonometric series that converge uniformly in $[-\pi, \pi]$ without converging absolutely in this interval.

In fact let us consider, for example, the series

$$\sum_{n=2}^{\infty} \frac{\sin nx}{n \ln n}. \tag{30.3}$$

Since $b_n = 1/(n \ln n)$, then $nb_n \to 0$ as $n \to \infty$ and moreover $b_n \downarrow 0$. This means that the given series converges uniformly in $[-\pi, \pi]$, but it does not converge absolutely in $[-\pi, \pi]$, since otherwise the series $\sum 1/(n \ln n)$ should converge and this series does actually diverge.†

We make this brief comment because very frequently in proving the uniform convergence of functional series Weierstrass's criterion (the comparison of the terms of a given series with the terms of a convergent numerical series) is used and in this case both absolute and uniform convergence is directly obtained.

In particular, for the trigonometric series $\sum b_n \sin nx$, where $\sum |b_n| < +\infty$, both absolute and uniform convergence occurs in $[-\pi, \pi]$, but in the example considered this is not so.

It is even possible to construct a trigonometric series which converges uniformly in $[-\pi, \pi]$ but which does not possess a single point of absolute convergence in this interval (see Chapter IX, § 3).

In connection with series of the type (30.2), where $b_n \downarrow 0$, it is useful to note yet another theorem:

THEOREM 3. *If $b_n \downarrow 0$ and the numbers nb_n are bounded, then the partial sums of the series*

$$\sum_{n=1}^{\infty} b_n \sin nx$$

are all bounded in $-\infty < x < +\infty$.

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† From the Lusin–Denjoy theorem which will be proved in § 61, it follows that the series (30.3) can converge absolutely only in a set of measure zero $\left(\text{because } \sum_{n=2}^{\infty} 1/(n \ln n) \text{ diverges}\right)$. Moreover, it is easily proved that the series (30.3) is not absolutely convergent at any $x \not\equiv 0 \pmod{\pi}$. Indeed, if for such x we had

$$\sum_{n=2}^{\infty} \frac{|\sin nx|}{n \ln n} < +\infty,$$

then

$$\sum_{n=2}^{\infty} \frac{\sin^2 nx}{n \ln n} < +\infty.$$

therefore

$$\sum_{n=2}^{\infty} \frac{(1 - \cos 2nx)}{n \ln n} < +\infty,$$

and since $\sum_{n=2}^{\infty} \frac{\cos 2nx}{n \ln n}$ converges, if $x \not\equiv 0 \pmod{\pi}$ then the convergence of $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ would follow and we would arrive at a contradiction.

Because of the periodicity and oddness of all the terms of the series it is sufficient to consider the interval $[0, \pi]$ and since at $x = 0$ and $x = \pi$ all the terms become zero, then we can confine ourselves to the case $0 < x < \pi$.

We have the condition

$$|kb_k| < M \quad (k = 1, 2, \ldots), \tag{30.4}$$

where M is a constant. Let us suppose

$$\nu = \left[\frac{\pi}{x}\right]. \tag{30.5}$$

If $n \leqslant \nu$, then

$$|S_n(x)| \leqslant \left|\sum_{k=1}^{n} b_k \sin kx\right| \leqslant \sum_{k=1}^{n} |kb_k|\, x \leqslant M x\nu \leqslant M\pi.$$

If $n > \nu$, then

$$S_n(x) = \sum_{k=1}^{\nu} b_k \sin kx + \sum_{\nu+1}^{n} b_k \sin kx = S_n^{(1)}(x) + S_n^{(2)}(x),$$

where $S_n^{(1)}(x)$ is estimated as in the preceding case, i.e.

$$|S_n^{(1)}(x)| \leqslant M\pi, \tag{30.6}$$

and to $S_n^{(2)}(x)$ we apply the corollary of Abel's transformation (see Introductory Material, § 1). Remembering (29.9)

$$|\bar{D}_n(x)| \leqslant \frac{\pi}{x} \quad \text{for} \quad 0 < |x| \leqslant \pi,$$

we find from (30.4) and (30.5)

$$|S_n^{(2)}(x)| \leqslant 2b_{\nu+1} \frac{\pi}{x} \leqslant 2M \frac{\pi}{x(\nu + 1)} \leqslant 2M. \tag{30.7}$$

From (30.6) and (30.7) it follows that

$$|S_n(x)| \leqslant M\pi + 2M = M(\pi + 2),$$

and Theorem 3 is proved.

COROLLARY. *We have for any n and x*

$$\left|\sum_{k=1}^{n} \frac{\sin kx}{k}\right| < C, \tag{30.8}$$

where C is an absolute constant.

Indeed, here we are concerned with the partial sums of the series

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n}, \tag{30.9}$$

in which $b_n = 1/n$, i.e. $b_n \downarrow 0$ and $nb_n = 1$.

Series (30.9) plays an important rôle in the many problems of the theory of trigonometric series; in § 41, in particular, we will investigate its behaviour in the neighbourhood of the point $x = 0$, since it permits us to obtain certain data on the behaviour of Fourier series for functions of bounded variation at those points where they are discontinuous.

In this paragraph we have considered only a few problems concerning sine and cosine series with monotonic coefficients. Chapter X will be devoted to a detailed investigation of this class of series. Here instead of referring the reader to Chapter X, we prove yet another important theorem concerning these series.

THEOREM 4. *If $a_n \downarrow 0$ and the sequence $\{a_n\}$ is convex, then the series*

$$\frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos jx \tag{30.1}$$

converges everywhere, apart from, perhaps, $x \equiv 0 \pmod{2\pi}$, to a non-negative summable function $f(x)$ and is the Fourier series for this function.

To prove, this, we consider

$$S_n(x) = \frac{a_0}{2} + \sum_{j=1}^{n} a_j \cos jx$$

and apply Abel's transformation; this gives

$$S_n(x) = \sum_{j=0}^{n-1} (a_j - a_{j+1}) D_j(x) + a_n D_n(x) = \sum_{j=0}^{n-1} \Delta a_j D_j(x) + a_n D_n(x), \tag{30.10}$$

where $\Delta a_j = a_j - a_{j+1}$. Supposing $\Delta^2 a_j = \Delta a_j - \Delta a_{j+1}$ and again using Abel's transformation, we find

$$S_n(x) = \sum_{j=0}^{n-2} \Delta^2 a_j \sum_{p=0}^{j} D_p(x) + \Delta a_{n-1} \sum_{p=0}^{n-1} D_p(x) + a_n D_n(x). \tag{30.11}$$

An expression of the form

$$K_j(x) = \frac{1}{j+1} \sum_{p=0}^{j} D_p(x) \tag{30.12}$$

is usually called *a Fejér kernel of order j*. We will study it in more detail in § 47. Here we shall refer to the fact that $K_j(x) \geqslant 0$ for all x (see (47.5)). From (30.11) and (30.12) it immediately follows that

$$S_n(x) = \sum_{j=0}^{n-2} (j+1) \Delta^2 a_j K_j(x) + n \Delta a_{n-1} K_{n-1}(x) + a_n D_n(x). \tag{30.13}$$

If $x \not\equiv 0 \pmod{2\pi}$, then since $a_n \to 0$ the last term of the right-hand side of (30.13) tends to zero as $n \to \infty$. Moreover, at $x \not\equiv 0 \pmod{2\pi}$ from (30.12) and (29.3) we note that $K_n(x)$ always remains finite as $n \to \infty$ and $n \Delta a_{n-1} \to 0$ for the convex sequences $\{a_n\}$ (see Introductory Material, § 3) and therefore $n \Delta a_{n-1} K_{n-1}(x) \to 0$ as $n \to \infty$. Hence for $x \not\equiv 0 \pmod{2\pi}$

$$f(x) = \lim_{n \to \infty} S_n(x) = \sum_{j=1}^{\infty} (j+1) \Delta^2 a_j K_j(x). \tag{30.14}$$

It is not necessary for us to prove the very existence of the limit, since the convergence of series (30.1) for all x, apart from $x \equiv 0 \pmod{2\pi}$, was established for $a_n \downarrow 0$, without the hypothesis of the convexity of $\{a_n\}$, in Theorem 1 of this section.

Thus, from (30.14) we conclude that the sum $f(x)$ of series (30.14) is a non-negative function, because all $\Delta^2 a_j \geqslant 0$ and $K_j(x) \geqslant 0$ for all x.

It remains for us to prove that the series (30.1) is a Fourier series of $f(x)$. For this purpose we note that because

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{30.15}$$

and because $a_n \downarrow 0$ the series on the right-hand side of (30.15) converges uniformly in (ε, π) for any $\varepsilon > 0$, then

$$\int_0^\pi f(x)\,dx = \frac{a_0}{2}\int_\varepsilon^\pi dx + \sum_{n=1}^{\infty} a_n \int_\varepsilon^\pi \cos nx\,dx = \frac{a_0}{2}(\pi - \varepsilon) - \sum_{n=1}^{\infty} a_n \frac{\sin n\varepsilon}{n}. \tag{30.16}$$

From $a_n \downarrow 0$ due to Theorem 2, it follows that the series $\sum (a_n \sin nx)/n$ converges uniformly in $[0, 2\pi]$, which means that its sum is continuous in this interval, and therefore the series on the right-hand side of (30.16) has a sum which tends to zero as $\varepsilon \to 0$. Hence it follows that

$$\lim_{\varepsilon \to 0} \int_\varepsilon^\pi f(x)\,dx = \frac{a_0}{2}\pi. \tag{30.17}$$

But since $f(x) \geqslant 0$, then from the existence of the limit on the left-hand side of (30.17) the summability of $f(x)$ in $[0, \pi]$ follows, and because $f(x)$ is even, this gives

$$\int_{-\pi}^{\pi} f(x)\,dx = 2\int_0^\pi f(x)\,dx = a_0\pi,$$

whence

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

We will now prove that at any $k = 1, 2, \ldots$ we have

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx.$$

For this purpose, multiplying both sides of (30.15) by $\cos kx$ and integrating in the interval $[\varepsilon, \pi]$, we find

$$\int_\varepsilon^\pi f(x)\cos kx\,dx = \frac{a_0}{2}\int_\varepsilon^\pi \cos kx\,dx + \sum_{n=1}^{k-1} a_n \int_\varepsilon^\pi \cos kx \cos nx\,dx$$

$$+ a_k \int_\varepsilon^\pi \cos^2 kx\,dx + \sum_{n=k+1}^{\infty} a_n \int_\varepsilon^\pi \cos kx \cos nx\,dx. \tag{30.18}$$

As $\varepsilon \to 0$ each of the integrals $\int_{\varepsilon}^{\pi} \cos kx\, dx$ and $\int_{\varepsilon}^{\pi} \cos kx \cos nx\, dx$ $(n = 1, 2, \dots k-1)$ tend to zero. Then

$$\lim_{\varepsilon \to 0} \int_{\varepsilon}^{\pi} \cos^2 kx\, dx = \int_{0}^{\pi} \cos^2 kx\, dx = \frac{\pi}{2}.$$

Finally

$$\sum_{n=k+1}^{\infty} a_n \int_{\varepsilon}^{\pi} \cos kx \cos nx\, dx = \sum_{n=k+1}^{\infty} a_n \int_{\varepsilon}^{\pi} \frac{\cos(k+n)x + \cos(n-k)x}{2}\, dx$$

$$= \sum_{n=k+1}^{\infty} a_n \left[\frac{\sin(k+n)\varepsilon}{2(k+n)} + \frac{\sin(n-k)\varepsilon}{2(n-k)} \right] \tag{30.19}$$

and arguing as previously, we see that as $\varepsilon \to 0$ the right-hand side of (30.19) tends to zero.

Thus, from (30.18) we obtain as $\varepsilon \to 0$

$$\int_{0}^{\pi} f(x) \cos kx\, dx = a_k \frac{\pi}{2}$$

and taking into account the evenness of $f(x)$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx\, dx.$$

Thus, series (30.1) is a Fourier series of $f(x)$ and the proof, therefore, is concluded.

COROLLARY. Since the sequence $1/\ln n$ $(n = 2, 3, \dots)$ is convex, then from the given theorem it follows in particular that: *the series*

$$\sum_{n=2}^{\infty} \frac{\cos nx}{\ln n} \tag{30.20}$$

is a Fourier series.

It is also known that $\sum_{n=2}^{\infty} (\sin nx)/\ln n$ is not a Fourier series (see § 40), therefore we see that a series conjugate to a Fourier series is not necessarily itself a Fourier series.

Note. It will be useful to us later to know that *the partial sums of series* (30.20) *satisfy the condition*

$$\int_{0}^{2\pi} |S_n(x)|\, dx < C, \tag{30.21}$$

where C is an absolute constant.

Indeed from formula (30.13) we obtain

$$\int_{0}^{2\pi} |S_n(x)|\, dx$$

$$\leqslant \sum_{j=0}^{n-2} (j+1) \Delta^2 a_j \int_{0}^{2\pi} K_j(x)\, dx + n \Delta a_{n-1} \int_{0}^{2\pi} K_{n-1}(x)\, dx + a_n \int_{0}^{2\pi} |D_n(x)|\, dx.$$

But since $\int_0^{2\pi} |D_n(x)|\,dx < A\ln n$, where A is a constant (see § 35) and

$$\int_0^{2\pi} K_j(x)dx = \frac{1}{j+1}\sum_{p=0}^{j}\int_0^{2\pi} D_p(x)\,dx = \pi,$$

then

$$\int_0^{2\pi} |S_n(x)|\,dx \leqslant \pi\left[\sum_{j=0}^{n-2}(j+1)\Delta^2 a_j + n\Delta a_{n-1}\right] + Aa_n\ln n.$$

This formula is true for any $a_n \downarrow 0$ forming a convex sequence. Therefore, taking into account that for such sequences $\sum_{j=0}^{\infty}(j+1)\Delta^2 a_j < +\infty$ (see Introductory Material, § 3) we have

$$\int_0^{2\pi} |S_n(x)|\,dx < Aa_n\ln n + B,$$

where A and B are constants. For the case we are considering when $a_n = 1/\ln n$, supposing that $A + B = C$, we see therefore that (30.21) is valid, i.e.

$$\int_0^{2\pi}\left|\sum_{k=1}^{n}\frac{\cos kx}{\ln k}\right|dx \leqslant C \quad (n = 1, 2, \ldots). \tag{30.22}$$

§ 31. Integral expressions for the partial sums of a Fourier series and its conjugate series

In order to study the question of the convergence of a Fourier series in the whole interval $[-\pi, \pi]$ or at any point of it, it seems very convenient to represent the partial sum of the series in the form given it by Dirichlet.

Let

$$\sigma(f) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k\cos kx + b_k\sin kx) \tag{31.1}$$

and

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k\cos kx + b_k\sin kx). \tag{31.2}$$

Substituting in (31.2) the expressions for a_k and b_k from the Fourier formulae, we find

$$S_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt + \sum_{k=1}^{n}\left[\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\,dt\right)\cos kx + \left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt\,dt\right)\sin kx\right]$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2} + \sum_{k=1}^{n}(\cos kt \cos kx + \sin kt \sin kx)\right] dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2} + \sum_{k=1}^{n}\cos k(t-x)\right] dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\, D_n(t-x)\, dt, \quad (31.3)$$

where $D_n(u)$ is a Dirichlet kernel (see § 29) and therefore

$$D_n(u) = \frac{\sin\left(n+\frac{1}{2}\right)u}{2\sin\frac{u}{2}}. \quad (31.4)$$

Supposing $t - x = u$, we obtain from (31.3) and (31.4)

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)\, D_n(u)\, du = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)\frac{\sin\left(n+\frac{1}{2}\right)u}{2\sin\frac{u}{2}}\, du. \quad (31.5)$$

If we have to consider simultaneously the Fourier series of several functions, for example, f, g, ψ, we will write $S_n(x,f)$, $S_n(x,g)$, $S_n(x,\psi)$ in order to distinguish between their partial sums. Using this notation, we note immediately that it follows directly from (31.5) that

$$\left.\begin{aligned} S_n(x, f_1 + f_2) &= S_n(x, f_1) + S_n(x, f_2), \\ S_n(x, Cf) &= CS_n(x, f), \end{aligned}\right\} \quad (31.6)$$

and if $f(x) = \sum_{k=1}^{\infty} f_k(x)$, where the series converges uniformly, then

$$S_n(x, f) = \sum_{k=1}^{\infty} S_n(x, f_k) \quad (31.7)$$

(because uniformly convergent series can be integrated term-by-term).

We will also note that since

$$|D_n(x)| \leqslant n + \frac{1}{2}$$

for any x, then in every case

$$|S_n(x, f)| \leqslant \left(n + \frac{1}{2}\right)\int_{0}^{2\pi} |f(t)|\, dt, \quad (31.8)$$

and although this estimate in the majority of cases is rough, it is, however, sometimes sufficient.

In investigating the problems of the convergence of formula (31.5), the series is usually transformed, but before going into this question, we will remark here that the partial sum of the series conjugate to (31.1) can be written in a similar manner, i.e.

$$\sum_{k=1}^{\infty}(-b_n\cos nx + a_n\sin nx).$$

Thus, supposing

$$\bar{S}_n(x) = \sum_{k=1}^{n}(-b_k\cos kx + a_k\sin kx),$$

we find following a similar argument

$$\bar{S}_n(x) = -\frac{1}{\pi}\int_0^{2\pi} f(t)\,\bar{D}_n(t-x)\,dt, \tag{31.9}$$

where

$$D_n(u) = \sum_{k=1}^{n}\sin ku.$$

The kernel $\bar{D}_n(u)$, conjugate to the Dirichlet kernel, as we have seen (see § 29), has the form

$$\bar{D}_n(x) = \frac{\cos\frac{u}{2} - \cos\left(n+\frac{1}{2}\right)u}{2\sin\frac{u}{2}}, \tag{31.10}$$

therefore

$$\bar{S}_n(x) = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,\frac{\cos\frac{t-x}{2} - \cos\left(n+\frac{1}{2}\right)(t-x)}{2\sin\frac{t-x}{2}}\,dt \tag{31.11}$$

or

$$\bar{S}_n(x) = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)\,\frac{\cos\frac{u}{2} - \cos\left(n+\frac{1}{2}\right)u}{2\sin\frac{u}{2}}\,du. \tag{31.12}$$

Now for the transformation of formulae (31.5) and (31.12) to more suitable forms, we will prove an important lemma.

LEMMA. *If $f(x)$ is summable, $g(x)$ bounded and both possess a period 2π, then the integrals*

$$\int_{-\pi}^{\pi} f(x+t)\,g(t)\cos nt\,dt \quad \textit{and} \quad \int_{-\pi}^{\pi} f(x+t)\,g(t)\sin nt\,dt \tag{31.13}$$

tend to zero uniformly as $n \to \infty$.

Proof. Let

$$\psi_x(t) = f(x+t)\,g(t).$$

If x is fixed, then $\psi_x(t)$ is a summable function of the variable t and therefore it is clear that the integrals being considered only differ by a constant multiplier $1/\pi$ from the Fourier coefficients of this function. Thus for every x the integrals (31.13) tend to zero as $n \to \infty$. But the significance of the lemma is to prove how uniformly they tend to zero.

Following the argument of § 21, we have

$$\left| \int_{-\pi}^{\pi} \psi_x(t) \cos nt\, dt \right| \leqslant \int_{-\pi}^{\pi} \left| \psi_x\left(t + \frac{\pi}{n}\right) - \psi_x(t) \right| dt$$

and similarly for $\sin nt$. Therefore it is sufficient to prove that

$$\int_{-\pi}^{\pi} \left| \psi_x\left(t + \frac{\pi}{n}\right) - \psi_x(t) \right| dt \tag{31.14}$$

tends to zero uniformly with respect to x as $n \to \infty$. But

$$\begin{aligned} \int_{-\pi}^{\pi} \left| \psi_x\left(t + \frac{\pi}{n}\right) - \psi_x(t) \right| dt &= \int_{-\pi}^{\pi} \left| f\left(x + t + \frac{\pi}{n}\right) g\left(t + \frac{\pi}{n}\right) - f(x+t)\, g(t) \right| dt \\ &\leqslant \int_{-\pi}^{\pi} \left| f\left(x + t + \frac{\pi}{n}\right) - f(x+t) \right| \left| g\left(t + \frac{\pi}{n}\right) \right| dt \\ &\quad + \int_{-\pi}^{\pi} \left| f(x+t) \right| \left| g\left(t + \frac{\pi}{n}\right) - g(t) \right| dt. \end{aligned} \tag{31.15}$$

Noting that $g(t)$ is bounded and has a period of 2π, then $|g(t)| \leqslant M$ for any t, and also remembering that $f(t)$ also has period 2π, we find for the first of the integrals on the right-hand side of (31.15)

$$\begin{aligned} &\int_{-\pi}^{\pi} \left| f\left(x + t + \frac{\pi}{n}\right) - f(x+t) \right| \left| g\left(t + \frac{\pi}{n}\right) \right| dt \\ &\leqslant M \int_{-\pi}^{\pi} \left| f\left(x + t + \frac{\pi}{n}\right) - f(x+t) \right| dt \leqslant M \int_{-\pi}^{\pi} \left| f\left(t + \frac{\pi}{n}\right) - f(t) \right| dt \\ &\leqslant M \omega_1\left(\frac{\pi}{n}, f\right), \end{aligned} \tag{31.16}$$

where $\omega_1(\delta, f)$ is the integral modulus of continuity of $f(x)$ (see Introductory Material, § 25); we already know that $\omega_1(\delta, f)$ tends to zero as $\delta \to 0$ for any summable $f(x)$. Since x no longer figures on the right-hand side of the inequality (31.16), we obtain

$$\int_{-\pi}^{\pi} \left| f\left(x + t + \frac{\pi}{n}\right) - f(x+t) \right| \left| g\left(t + \frac{\pi}{n}\right) \right| dt \to 0$$

uniformly relative to x as $n \to \infty$.

For the estimate of the second integral of formula (31.15), we take any $\varepsilon > 0$ and resolve $f(x)$ into the sum of two functions $f_1(x)$ and $f_2(x)$, of which the first is bounded, for example, $|f_1(x)| \leqslant K$ and for the second

$$\int_{-\pi}^{\pi} |f_2(t)|\, dt < \varepsilon.$$

Then

$$\int_{-\pi}^{\pi} \left| f(x+t) \right| \left| g\left(t + \frac{\pi}{n}\right) - g(t) \right| dt$$

$$\leqslant K \int_{-\pi}^{\pi} \left| g\left(t + \frac{\pi}{n}\right) - g(t) \right| dt + \int_{-\pi}^{\pi} \left| f_2(x+t) \right| \left| g\left(t + \frac{\pi}{n}\right) - g(t) \right| dt$$

$$\leqslant K\omega_1\left(\frac{\pi}{n}, g\right) + 2M \int_{-\pi}^{\pi} \left| f_2(x+t) \right| dt \leqslant K\omega_1\left(\frac{\pi}{n}, g\right) + 2M\varepsilon. \tag{31.17}$$

Since $\omega_1(\pi/n, g) \to 0$, the number ε is arbitrary and x does not enter into the right-hand side of (31.17), then the left-hand side of (31.17) tends to zero uniformly and the proof is concluded.

Note 1. Our lemma holds if instead of the integrals of (31.13) we consider the integrals

$$\int_a^b f(x+t) g(t) \cos nt\, dt \quad \text{and} \quad \int_a^b f(x+t) g(t) \sin nt\, dt,$$

where a and b are any two points in $[-\pi, \pi]$. Indeed, it is sufficient to assume that

$$g_1(t) = \begin{cases} g(t) & \text{in} \quad [a, b], \\ 0 & \text{outside } [a, b], \end{cases}$$

which would reduce this case to the preceding one.

Note 2. In carrying out the proof we never made use of the fact that n is an integer. Therefore the lemma holds if $n \to \infty$ passing through all real values.

Note 3. It will be useful later to know that our lemma holds if instead of $g(t)$ we consider the function $g_x(t)$ for which the following conditions are fulfilled

(a) $|g_x(t)| \leqslant M$ for $\begin{array}{l} -\pi \leqslant x \leqslant \pi, \\ -\pi \leqslant t \leqslant \pi \end{array}$

and moreover as $h \to 0$

(b) $\int_{-\pi}^{\pi} |g_x(t+h) - g_x(t)|\, dt \to 0$

uniformly relative to x in $[-\pi, \pi]$.

Indeed in this case the proof of the lemma is exactly the same.

Note 4. If $f(x)$ is a continuous function, then from the given proof we obtain

$$\left|\int_{-\pi}^{\pi} f(x+t)g(t)\cos nt\, dt\right| \leqslant A\omega\left(\frac{\pi}{n}, f\right) + B\omega_1\left(\frac{\pi}{n}, g\right),$$

$$\left|\int_{-\pi}^{\pi} f(x+t)g(t)\sin nt\, dt\right| \leqslant A\omega\left(\frac{\pi}{n}, f\right) + B\omega_1\left(\frac{\pi}{n}, g\right),$$

where $\omega(\delta, f)$ is the modulus of continuity of $f(x)$ and A and B are constants.

In fact, in formula (31.16) when $f(x)$ is continuous, $2\pi\omega(\pi/n, f)$ can be substituted for $\omega_1(\pi/n, f)$, and as $f(x)$ is bounded, the second integral of formula (31.15) does not exceed $B\omega_1(\pi/n, g)$ where B is a constant.

§ 32. Simplification of expressions for $S_n(x)$ and $\bar{S}_n(x)$

We will now use the lemma proved in § 31 to simplify the expressions for $S_n(x)$ and $\bar{S}_n(x)$ (see (31.5) and (31.11)).

We will first note that

$$\frac{\sin\left(n+\frac{1}{2}\right)u}{2\sin\frac{u}{2}} = \frac{\sin nu \cos\frac{u}{2} + \cos nu \sin\frac{u}{2}}{2\sin\frac{u}{2}} = \frac{\sin nu}{2\operatorname{tg}\frac{u}{2}} + \frac{1}{2}\cos nu. \quad (32.1)\dagger$$

We will also note that the function

$$g(u) = \frac{1}{2\operatorname{tg}\frac{u}{2}} - \frac{1}{u} \qquad (32.2)$$

is continuous in $[-\pi, \pi]$. The only uncertainty could be caused by the point $u = 0$; but, using L'Hôpital's rule, it is easily seen that $\lim_{u\to 0} g(u) = 0$. We still require that $g(u + 2\pi) = g(u)$; then $g(u)$ is bounded in $(-\infty, +\infty)$.

From (32.1) and (32.2) we obtain

$$\frac{\sin\left(n+\frac{1}{2}\right)u}{2\sin\frac{u}{2}} = \frac{\sin nu}{u} + g(u)\sin nu + \frac{1}{2}\cos nu. \qquad (32.3)$$

† The Continental abbreviation "tg" is used for "tangent" throughout this work (Translator).

Therefore from (31.5) we obtain

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)\frac{\sin nu}{u}\,du + \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)g(u)\sin nu\,du$$

$$+ \frac{1}{2\pi}\int_{-\pi}^{\pi} f(u+x)\cos nu\,du. \tag{32.4}$$

The last two integrals of formula (32.4) tend to zero uniformly as $n \to \infty$ due to the lemma of § 31 and since $g(u)$ is bounded. Therefore

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)\frac{\sin nu}{u}\,du + o(1), \tag{32.5}$$

where $o(1)$ is a magnitude which tends to zero uniformly. We will frequently use this fact.

Note. It is sometimes important to estimate the magnitude $o(1)$ more exactly; therefore we will now show that if $f(x)$ is continuous, then from Note 4 made at the end of § 31, the modulus of each of the last two integrals in (32.4) does not exceed

$$A\omega\left(\frac{\pi}{n}, f\right) + B\omega_1\left(\frac{\pi}{n}, g\right), \tag{32.6}$$

where A and B are constants. But since $g(u)$ is a function of bounded variation and for such functions the integral modulus of continuity $\omega_1(\delta)$ has the order $O(\delta)$ (see Introductory Material, § 25), then (32.6) is a magnitude of order

$$O\left[\omega\left(\frac{\pi}{n}, f\right)\right] + O\left(\frac{1}{n}\right). \tag{32.7}$$

Finally, having noted that for any continuous function $f(x)$ the modulus of continuity $\omega(\delta, f)$ does not exceed $O(\delta)$, we conclude that in (32.7) the second term is either of the same order as the first or is infinitely small of a higher order. Therefore finally, supposing that

$$\tilde{S}_n(x, f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)\frac{\sin nu}{u}\,du, \tag{32.8}$$

we find from (32.5) for continuous $f(x)$

$$S_n(x, f) = \tilde{S}_n(x, f) + O\left[\omega\left(\frac{\pi}{n}, f\right)\right]. \tag{32.9}$$

When $f(x)$ is any summable function, it is sometimes useful to estimate

$$|S_n(x, f) - \tilde{S}_n(x, f)| \leqslant C\int_0^{2\pi} |f(x)|\,dx, \tag{32.10}$$

where C is an absolute constant. This estimate is obtained directly from (32.4) and (32.8) if it is remembered that the function $g(u)$ is bounded.

After this remark which will be useful later, let us return to the simplification of formulae for partial sums. We wish to simplify the expression for $\bar{S}_n(x)$. For this purpose we note that (see (31.10))

$$\bar{D}_n(u) = \frac{\cos\frac{u}{2} - \cos\left(n+\frac{1}{2}\right)u}{2\sin\frac{u}{2}} = \frac{\cos\frac{u}{2} - \cos\frac{u}{2}\cos nu}{2\sin\frac{u}{2}} + \frac{\sin nu}{2}$$

$$= \frac{1-\cos nu}{2\operatorname{tg}\frac{u}{2}} + \frac{\sin nu}{2}. \qquad (32.11)$$

Hence if the lemma of § 31 is used, we immediately obtain

$$\bar{S}_n(x) = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)\frac{1-\cos nu}{2\operatorname{tg}\frac{u}{2}}\,du + o(1).$$

If the function $g(u)$ is used again, then another expression can be obtained for $\bar{S}_n(x)$. Namely, if we write

$$\bar{D}_n(u) = \frac{1-\cos nu}{u} + g(u)(1-\cos nu) + \frac{\sin nu}{2},$$

then, again using the lemma of § 31 we obtain

$$\bar{S}_n(x) = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)\frac{1-\cos nu}{u}\,du - \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)g(u)\,du + o(1),$$

and since the second integral is $O(1)$, then

$$\bar{S}_n(x) = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)\frac{1-\cos nu}{u}\,du + O(1) \qquad (32.12)$$

or

$$\bar{S}_n(x) = -\frac{1}{\pi}\int_{0}^{\pi} [f(x+u)-f(x-u)]\frac{1-\cos nu}{u}\,du + O(1). \qquad (32.13)$$

For future reference it will also be useful to note that, if $\delta > 0$ is arbitrary, and $f(x)$ is bounded, then it is possible to rewrite (32.13) in the form

$$\bar{S}_n(x) = -\frac{1}{\pi}\int_{0}^{\delta} [f(x+u)-f(x-u)]\frac{1-\cos nu}{u}\,du + O(1), \qquad (32.14)$$

since the discarded integral $\int_{\delta}^{\pi}$ is $O(1)$.

§ 33. Riemann's principle of localization

In § 32 we found a suitable expression for the partial sum of the Fourier series, from which an important corollary can easily be drawn. First, taking any $\delta > 0$ and denoting by $g(u)$ a function, defined thus

$$g(u) = \begin{cases} 0 & \text{in } (-\delta, \delta), \\ \dfrac{1}{u} & \text{in } (-\pi, -\delta) \text{ and } (\delta, \pi), \end{cases}$$

$$g(u + 2\pi) = g(u),$$

on the basis of (32.5) we can write

$$S_n(x) = \frac{1}{\pi} \int_{-\delta}^{\delta} f(u + x) \frac{\sin nu}{u} \, du + \frac{1}{\pi} \int_{-\pi}^{\pi} f(u + x) g(u) \sin nu \, du + o(1),$$

and since $g(u)$ is bounded and periodic, it follows that

$$S_n(x) = \frac{1}{\pi} \int_{-\delta}^{\delta} f(u + x) \frac{\sin nu}{u} \, du + o(1), \tag{33.1}$$

where again $o(1)$ tends to zero.† This formula allows the following extremely important theorem, known as *Riemann's principle of localization*, to be expressed.

RIEMANN'S THEOREM. *The convergence or divergence of a Fourier series at a point x depends only on the behaviour of the function $f(x)$ in the neighbourhood of the point x.*

In fact, the value of the function $f(x)$ outside the interval $(x - \delta, x + \delta)$ does not figure at all in formula (33.1), and therefore the question whether $S_n(x)$ tends to a limit as $n \to \infty$ depends only on the behaviour of $f(x)$ in this interval. Moreover, since in formula (33.1), as has already been proved, $o(1)$ tends uniformly to zero, it is possible to judge the uniform convergence of $S_n(x)$ in any interval by whether the integral on the right-hand side of (33.1) tends uniformly to a limit.

This result is appropriately expressed in the form:

THEOREM. *If two functions $f_1(x)$ and $f_2(x)$ coincide in some interval $[a, b]$, then in any interval $[a + \varepsilon, b - \varepsilon]$ where $\varepsilon > 0$, their Fourier series are uniformly equiconvergent, i.e. the difference of these series converges uniformly to zero.*

Indeed, let

$$f(x) = f_1(x) - f_2(x).$$

† We draw the reader's attention to the work by Hille and Klein[1] where it is proved that

$$\left| S_n(x, f) - \frac{1}{\pi} \int_{-\delta}^{\delta} f(x + t) \frac{\sin nt}{t} \, dt \right| \leqslant \frac{K}{\delta} \left[\int_0^{2\pi} |f(x)| \, dx + 1 \right] \omega_1 \left(\frac{1}{n}, f \right).$$

Here $\omega_1(\delta, f)$ is the integral modulus of continuity of $f(x)$ and K is an absolute constant.

Then $f(x) = 0$ in $[a, b]$. Let a number $\delta > 0$ be chosen such that $\delta \leqslant \varepsilon$ and x is any point of the interval $[a+\varepsilon, b-\varepsilon]$. Then $u + x \in [a, b]$ for $-\delta \leqslant u \leqslant \delta$ and therefore $f(u+x) = 0$; from formula (33.1):

$$S_n(x) = o(1) \quad \text{in} \quad [a+\varepsilon, b-\varepsilon],$$

where $o(1)$ tends uniformly to zero in $[0, 2\pi]$. This means that the Fourier series of $f(x)$ converges uniformly to zero in $[a+\varepsilon, b-\varepsilon]$.

§ 34. Steinhaus's theorem

A useful corollary can be derived from the preceding results. It is due to Steinhaus[3] and can be expressed in the following form:

If $\lambda(x)$ is a periodic function, satisfying Lipschitz' condition of order 1, then the series $\sigma(\lambda f)$ and $\lambda(x)\sigma(f)$ are uniformly convergent in $[-\pi, \pi]$.

In fact, we have

$$S_n(\lambda f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\lambda(x+t)\frac{\sin nt}{t}\,dt + o(1),$$

$$\lambda(x)S_n(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\lambda(x)\frac{\sin nt}{t}\,dt + o(1).$$

Therefore, supposing

$$g_x(t) = \frac{\lambda(x+t) - \lambda(x)}{t},$$

we have

$$S_n(\lambda f) - \lambda(x)S_n(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)g_x(t)\sin nt\,dt + o(1). \tag{34.1}$$

In order to prove that the right-hand side of (34.1) tends uniformly to zero, it is sufficient to apply to it the first part of Note 3 of the lemma in §31, providing only that the limitations imposed there on $g_x(t)$ are fulfilled. But the condition

$$|g_x(t)| \leqslant M$$

uniformly with respect to x and t is the result of the fact that $g_x(t)$ satisfies the Lipschitz condition of order 1; it remains to prove that

$$\int_{-\pi}^{\pi} |g_x(t+h) - g_x(t)|\,dt = o(1)$$

uniformly relative to x as $h \to 0$.

To do so, taking $\varepsilon > 0$, we will consider an interval of length $(-\varepsilon, \varepsilon)$; in it we have

$$\int_{-\varepsilon}^{\varepsilon} |g_x(t+h) - g_x(t)|\,dt \leqslant 4M\varepsilon.$$

If $t \in (-\pi, -\varepsilon)$ or $t \in (\varepsilon, \pi)$, then for any η it is possible to find h such that the expression under the integral sign for all t in the considered interval will be less than η, and therefore the corresponding integral less than $\pi\eta$. This concludes the proof of the theorem.

§ 35. Integral $\int\limits_0^\infty \frac{\sin x}{x}\,dx$. Lebesgue constants

Before continuing with the study of the convergence of a Fourier series, we should mention certain properties of the expression

$$D_n^*(t) = \frac{\sin nt}{t}, \tag{35.1}$$

which we will call *a simplified Dirichlet kernel.* Let us note first that from formula (33.1), taking into account the evenness of the simplified Dirichlet kernel, we immediately find that

$$S_n(x) = \frac{1}{\pi}\int\limits_0^\delta [f(x+u) + f(x-u)]\,\frac{\sin nu}{u}\,du + o(1). \tag{35.2}$$

If we consider the case $f(x) \equiv 1$, then $S_n(x) \equiv 1$ for any n, and therefore

$$1 = \frac{2}{\pi}\int\limits_0^\delta \frac{\sin nu}{u}\,du + o(1). \tag{35.3}$$

Supposing $nu = t$, we then find that

$$1 = \frac{2}{\pi}\int\limits_0^{n\delta} \frac{\sin t}{t}\,dt + o(1),$$

and therefore

$$\lim_{n\to\infty}\int\limits_0^{n\delta} \frac{\sin t}{t}\,dt = \frac{\pi}{2}.$$

Hence it immediately follows that

$$\int\limits_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}, \tag{35.4}$$

i.e. this improper integral has meaning and we even know its magnitude.

From the existence of this integral it follows that: if $\delta > 0$ and $\delta' > 0$, then

$$\lim_{n\to\infty}\int\limits_\delta^{\delta'} \frac{\sin nt}{t}\,dt = \lim_{n\to\infty}\int\limits_{n\delta}^{n\delta'} \frac{\sin t}{t}\,dt = 0. \tag{35.5}$$

This formula will be necessary later.

We note that it is necessary to understand the existence of the integral (35.4) geometrically; therefore we will dwell somewhat on this problem, although it should be known to the reader from courses on analysis.

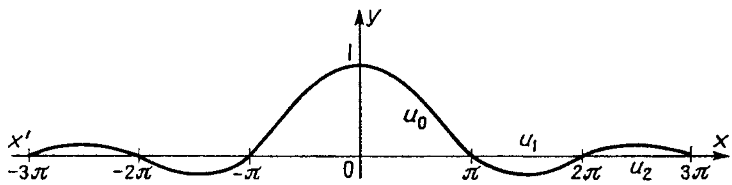

FIG. 5

The convergence of the integral $\int_0^\infty [(\sin t)/t]\, dt$ could also be proved in another way. Supposing (Fig. 5)

$$u_k = \int_{k\pi}^{(k+1)\pi} \frac{\sin t}{t}\, dt \quad (k = 0, 1, 2, \ldots),$$

we see that

$$u_k = \int_0^\pi \frac{\sin(t + k\pi)}{t + k\pi}\, dt = (-1)^k \int_0^\pi \frac{\sin t}{t + k\pi}\, dt,$$

whence it follows that the series $\sum_{k=0}^\infty u_k$ alternates its signs, whilst its terms monotonically decrease in their absolute value and tend to zero, since

$$|u_k| = \int_0^\pi \frac{\sin t}{t + k\pi}\, dt < \frac{1}{k\pi}\pi = \frac{1}{k} \quad (k = 1, 2, \ldots).$$

But according to Leibniz' well-known theorem this type of series should converge. On the other hand it is clear that when the sum $\sum u_k$ has meaning, then it is the integral (35.4). Thus,

$$\sum_{k=0}^\infty u_k = \int_0^\infty \frac{\sin u}{u}\, du = \frac{\pi}{2}.$$

We now note that

$$\sum_{k=0}^\infty u_k < u_0,$$

whence

$$\frac{\pi}{2} < u_0 = \int_0^\pi \frac{\sin u}{u}\, du < \pi. \tag{35.6}$$

Thus from the monotonic nature of u_n and the alternating of their signs, we see that if A and B are any two numbers such that $0 \leqslant A < B$, then

$$\left| \int_A^B \frac{\sin t}{t}\, dt \right| < \pi. \tag{35.7}$$

Because of the evenness of $(\sin t)/t$, this is also true if $A < B \leqslant 0$.

Finally, if A and B are of different signs, then dividing the integral by two, namely from A to 0 and from 0 to B, we find that

$$\left| \int_A^B \frac{\sin t}{t}\, dt \right| < 2\pi.$$

This simple statement will be very important to us later, since it follows from it that for any a and b we have

$$\left| \int_a^b \frac{\sin nt}{t}\, dt \right| < 2\pi, \tag{35.8}$$

because

$$\left| \int_a^b \frac{\sin nt}{t}\, dt \right| = \left| \int_{na}^{nb} \frac{\sin t}{t}\, dt \right| < 2\pi \tag{35.9}$$

by virtue of (35.7).

Now the fact that the integral of (35.8) is bounded is solely due to the interference of the positive and negative sinusoidal waves. If the modulus of the expression under the integral sign were taken, then the result would be completely different. We will prove that

$$\int_0^\pi \left| \frac{\sin nt}{t} \right| dt$$

increases without bound on increase of n and we will even estimate the order of its growth exactly. This will be very important later.

Let

$$I_n = \int_0^\pi \left| \frac{\sin nt}{t} \right| dt = \int_0^{n\pi} \left| \frac{\sin u}{u} \right| du. \tag{35.10}$$

Then it is clear that

$$I_{n+1} - I_n = \int_{n\pi}^{(n+1)\pi} \left| \frac{\sin u}{u} \right| du = \int_0^\pi \frac{\sin v}{v + n\pi}\, dv$$

and since at $0 \leqslant v \leqslant \pi$, we have

$$\frac{1}{(n+1)\pi} \leqslant \frac{1}{v + n\pi} \leqslant \frac{1}{n\pi} \quad (n = 1, 2, \ldots),$$

and

$$\int_0^\pi \sin v\, dv = 2,$$

then

$$\frac{2}{\pi(n+1)} \leqslant I_{n+1} - I_n \leqslant \frac{2}{n\pi}. \tag{35.11}$$

Letting n run through the values $1, 2, \ldots, m-1$ and summing the inequalities (35.11) we find

$$\frac{2}{\pi}\sum_{n=1}^{m-1}\frac{1}{n+1} \leqslant \sum_{n=1}^{m-1}(I_{n+1}-I_n) \leqslant \frac{2}{\pi}\sum_{n=1}^{m-1}\frac{1}{n}$$

or

$$I_1 + \frac{2}{\pi}\sum_{n=2}^{m}\frac{1}{n} \leqslant I_m \leqslant I_1 + \frac{2}{\pi}\sum_{n=1}^{m-1}\frac{1}{n}.$$

But taking into account that

$$1 + \frac{1}{2} + \cdots + \frac{1}{m} \approx \ln m,$$

where $\approx$ denotes an asymptotic equality (see Introductory Material, § 11), we find $I_m \approx \ln m$. Thus, we find

$$\int_0^{n\pi}\frac{|\sin u|}{u}\,du = I_n \approx \frac{2}{\pi}\ln n. \tag{35.12}$$

Thus, I_n increases infinitely with increase of n, and we also see the exact order of this increase.

It immediately follows from (35.12) that

$$\lim_{n\to\infty}\int_0^{n\pi}\left|\frac{\sin u}{u}\right|du = +\infty,$$

i.e. the integral

$$\int_0^{\infty}\left|\frac{\sin x}{x}\right|dx = \infty, \tag{35.13}$$

which means that the integral (35.4) is known to converge only conditionally, not absolutely.

From formula (35.12) we will derive a corollary, which will play an important role later.

A Lebesgue constant is defined by the expression

$$L_n = \frac{1}{\pi}\int_{-\pi}^{\pi}|D_n(t)|\,dt, \tag{35.14}$$

where $D_n(t)$ is a Dirichlet kernel.

Since $D_n(t)$ is an even function,

$$L_n = \frac{2}{\pi}\int_0^{\pi}|D_n(t)|\,dt.$$

But we know (see § 32) that

$$D_n(t) = \frac{\sin nt}{t} + O(1),$$

and therefore

$$L_n = \frac{2}{\pi} \int_0^{\pi} \left| \frac{\sin nt}{t} \right| dt + O(1),$$

whence

$$L_n = \frac{2}{\pi} I_n + O(1)$$

and due to (35.12)

$$L_n \approx \frac{4}{\pi^2} \ln n.$$

Thus

$$L_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |D_n(t)|\, dt \approx \frac{4}{\pi^2} \ln n. \tag{35.15}$$

It can be proved similarly that for a kernel conjugate to a Dirichlet kernel the integral of the modulus has the same order of increase, i.e. it increases as $\ln n$.

To prove this, we will consider an auxiliary integral, namely

$$J_n = \int_0^{\pi/2} \frac{\sin^2 nt}{\sin t}\, dt. \tag{35.16}$$

Since

$$\frac{\sin^2 nt}{\sin t} = \sum_{k=1}^{n} \sin(2k-1)t$$

(which is proved directly by multiplying both sides by $\sin t$ and changing the product of sines to the difference of cosines), then

$$J_n = \sum_{k=1}^{n} \int_0^{\pi/2} \sin(2k-1)t\, dt = \sum_{k=1}^{n} \frac{1}{2k-1} \sim \ln n \tag{35.17}$$

(here and later, we will not calculate the constant exactly but will simply write $u_n \sim v_n$, if $A < u_n/v_n < B$, where A and B are positive constants).

Let us now consider

$$\varrho_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |\bar{D}_n(t)|\, dt = \frac{2}{\pi} \int_0^{\pi} |\bar{D}_n(t)|\, dt.$$

Since (see (32.11))

$$\bar{D}_n(t) = \frac{1 - \cos nt}{2 \operatorname{tg} \frac{t}{2}} + \frac{\sin nt}{2},$$

then we have

$$\bar{D}_n(t) = \frac{1 - \cos nt}{2 \sin \frac{t}{2}} + O(1) = \frac{\sin^2 \frac{n}{2} t}{\sin \frac{t}{2}} + O(1)$$

$$\left(\text{because} \quad \frac{1}{2\,\mathrm{tg}\,\frac{t}{2}} - \frac{1}{2\sin\frac{t}{2}} = \frac{1}{2\sin\frac{t}{2}}\left[\cos\frac{t}{2} - 1\right] = -\frac{1}{2}\,\mathrm{tg}\,\frac{t}{4}\right).$$

Therefore

$$\varrho_n = \frac{2}{\pi}\int_0^{\pi} \frac{\sin^2 n\frac{t}{2}}{\sin\frac{t}{2}}\,dt + O(1) = \frac{4}{\pi}\int_0^{\pi/2} \frac{\sin^2 nu}{\sin u}\,du + O(1) = \frac{4}{\pi}J_n + O(1),$$

and therefore

$$\varrho_n \sim \ln n.$$

Thus,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \left|\bar{D}_n(t)\right|\,dt \sim \ln n, \tag{35.18}$$

and this is what we wanted to prove.

§ 36. Estimate of the partial sums of a Fourier series of a bounded function

From the results of the preceding section we immediately obtain the following theorem:

Lebesgue's Theorem. *If $f(x)$ is a bounded function*

$$|f(x)| \leqslant M,$$

then for $n = 2, 3, \ldots$

$$|S_n(x)| \leqslant CM \ln n, \quad 0 \leqslant x \leqslant 2\pi \tag{36.1}$$

and

$$\left|\bar{S}_n(x)\right| \leqslant CM \ln n, \quad 0 \leqslant x \leqslant 2\pi, \tag{36.2}$$

where C is an absolute constant.

Indeed (see (31.3) and (35.15)),

$$|S_n(x)| = \left|\frac{1}{\pi}\int_{-\pi}^{\pi} f(t) D_n(t-x)\,dt\right| \leqslant M\frac{1}{\pi}\int_{-\pi}^{\pi} |D_n(t-x)|\,dt = ML_n < CM\ln n$$

and similarly (see (31.9) and (35.18))

$$|\bar{S}_n(x)| < CM \ln n.$$

The theorem is proved.

Note 1. It could be thought that formula (36.1) is extremely rough; in fact, it can be proved that for a bounded function the partial sums of a Fourier series should be bounded. However, this is untrue even for continuous functions. If the Fourier series of a continuous function converged uniformly towards it, then such a bound should occur; but we will see later that for continuous functions Fourier series can converge non-uniformly, and can also diverge and even have unbounded partial sums in an infinite set of points.

Note 2. If $f(x) \in L[0, 2\pi]$ and $|f(x)| \leqslant M$ in some $[a, b] \subset [0, 2\pi]$, then in any $[a', b']$, $a < a' < b' < b$, we have

$$|S_n(x)| \leqslant AM \ln n + \frac{1}{\delta} \int_{-\pi}^{\pi} |f(t)|\, dt \quad (n = 2, 3, \ldots), \tag{36.3}$$

where A is an absolute constant and $\delta = \min(a' - a, b - b')$.

Indeed, since

$$S_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t + x) D_n(t)\, dt$$

$$= \frac{1}{\pi} \int_{-\delta}^{\delta} f(t + x) D_n(t)\, dt + \frac{1}{\pi} \int_{[-\pi, \pi] - (-\delta, \delta)} f(x + t) D_n(t)\, dt, \tag{36.4}$$

then by choosing δ so that $\delta = \min(a' - a,\ b - b')$, we see that at $x \in [a', b']$ the argument $t + x$ in the first integral does not go outside $[a, b]$ which means that

$$\left| \frac{1}{\pi} \int_{-\delta}^{\delta} f(t + x) D_n(t)\, dt \right| \leqslant M \frac{1}{\pi} \int_{-\pi}^{\pi} |D_n(t)|\, dt \leqslant AM \ln n, \tag{36.5}$$

where A is an absolute constant.

But since outside $(-\delta, \delta)$ we have $|D_n(t)| \leqslant \pi/\delta$ then for the second integral in (36.4) we find

$$\left| \frac{1}{\pi} \int_{[-\pi, \pi] - (-\delta, \delta)} f(x + t) D_n(t)\, dt \right| \leqslant \frac{1}{\delta} \int_{-\pi}^{\pi} |f(t)|\, dt. \tag{36.6}$$

Combining (36.4), (36.5) and (36.6), we obtain (36.3). Instead of (36.3) we can also write

$$|S_n(x)| \leqslant CM \ln n \quad \text{as} \quad n \geqslant N,$$

where N varies with M, δ and $\int_{-\pi}^{\pi} |f(t)|\, dt$, since if N is sufficiently large, then at $n \geqslant N$ the second term of formula (36.3) becomes less than the first.

§ 37. Criterion of convergence of a Fourier series

Let us return to the problem of the convergence of Fourier series. We want to find the conditions under which $\sigma(f)$ converges at some point x to some value S.

For this purpose we first remark that it follows from (33.1) that

$$S_n(x) = \frac{1}{\pi} \int_{0}^{\delta} [f(x + t) + f(x - t)] \frac{\sin nt}{t}\, dt + o(1), \tag{37.1}$$

where $o(1)$ signifies a magnitude tending uniformly to zero in $[-\pi, \pi]$. Moreover, multiplying both sides of equality (35.3) by S we have

$$S = \frac{2}{\pi} \int_0^\delta S \frac{\sin nu}{u} du + o(1). \tag{37.2}$$

From (37.1) and (37.2) we now find that

$$S_n(x) - S = \frac{1}{\pi} \int_0^\delta [f(x+u) + f(x-u) - 2S] \frac{\sin nu}{u} du + o(1). \tag{37.3}$$

It is clear from this that for the convergence of $\sigma(f)$ to the value S at the point x it is necessary and sufficient that

$$\lim_{n\to\infty} \int_0^\delta [f(x+u) + f(x-u) - 2S] \frac{\sin nu}{u} du = 0. \tag{37.4}$$

If we wish the series $\sigma(f)$ to have a "natural sum" at the point x, i.e. a sum equal to $f(x)$, then it is necessary and sufficient that

$$\lim_{n\to\infty} \int_0^\delta [f(x+u) + f(x-u) - 2f(x)] \frac{\sin nu}{u} du = 0. \tag{37.5}$$

Supposing

$$\varphi_x(u) = f(x+u) + f(x-u) - 2f(x), \tag{37.6}$$

we can therefore formulate this statement:

For the series $\sigma(f)$ to converge to $f(x)$ at some point x, it is necessary and sufficient that

$$\lim_{n\to\infty} \int_0^\delta \varphi_x(u) \frac{\sin nu}{u} du = 0, \tag{37.7}$$

where $\delta > 0$ and $\varphi_x(u)$ is defined by formula (37.6).

If the function $f(x)$ is continuous in some interval (a, b), then it is possible to raise the question of the uniform convergence of the series $\sigma(f)$ to $f(x)$.

Given any $\varepsilon > 0$. From the continuity of $f(x)$ in the interval (a, b), it follows that it is continuous and therefore bounded in the interval $[a+\varepsilon, b-\varepsilon]$. Therefore, if (35.3) is multiplied by $f(x)$, we have

$$f(x) = \frac{2}{\pi} \int_0^\delta f(x) \frac{\sin nt}{t} dt + o(1), \tag{37.8}$$

where $o(1)$ tends uniformly to zero in $[a+\varepsilon, b-\varepsilon]$. From (37.1) and (37.8) we then derive

$$S_n(x) - f(x) = \frac{1}{\pi} \int_0^\delta [f(x+u) + f(x-u) - 2f(x)] \frac{\sin nu}{u} du + o(1). \tag{37.9}$$

Here δ can be taken as any number. Therefore if we take $\delta < \varepsilon$, then for $x \in [a + \varepsilon, b - \varepsilon]$ and $|u| \leqslant \delta$ we will have $u + x \in (a, b)$ and $u - x \in (a, b)$, and then $\varphi_x(u)$, defined by formula (37.6), will be still more continuous in (a, b). Hence, using (37.9), we can conclude:

If $f(x)$ is continuous in (a, b) and given any $\varepsilon > 0$, then for uniform convergence of the series $\sigma(f)$ in $[a + \varepsilon, b - \varepsilon]$, it is necessary and sufficient that

$$\lim_{n\to\infty} \int_0^\delta \varphi_x(u) \frac{\sin nu}{u} du = 0$$

uniformly in $[a, b]$; here δ is any number satisfying the inequality $0 < \delta < \varepsilon$, and $\varphi_x(u)$ is a function defined by the equality (37.6) and continuous for

$$a + \varepsilon \leqslant x \leqslant b - \varepsilon, \quad |u| \leqslant \delta.$$

§ 38. Dini's test

The conditions that have been obtained for convergence (and for uniform convergence), even though they are necessary and sufficient, are very difficult to apply. Therefore we derive from them a series of tests which will be sufficient only for convergence (or for uniform convergence) but are frequently found to be very useful in simple and important cases.

Before deriving these tests, we will give a definition.

DEFINITION. Following Lebesgue, we say that the point x_0 is *regular*, if $f(x_0 - 0)$ and $f(x_0 + 0)$ exist and if

$$f(x_0) = \frac{f(x_0 + 0) + f(x_0 - 0)}{2}.$$

It is clear that any point of continuity is regular; also the points of discontinuity of the first kind will be regular, that is, those in which the magnitude of the function is the arithmetic mean of its limits left and right.

Let us prove the following theorem:

Dini's test. The series $\sigma(f)$ converges to $f(x)$ at every regular point x, where the integral

$$\int_0^\delta |f(x + u) + f(x - u) - 2f(x)| \frac{du}{u}$$

has meaning.

Indeed, if this integral has meaning, then it is possible for any $\varepsilon > 0$ to choose η so small that

$$\int_0^\eta |f(x + u) + f(x - u) - 2f(x)| \frac{du}{u} < \varepsilon.$$

Then for any n, since $|\sin nu| \leqslant 1$, we have

$$\left| \int_0^\eta [f(x + u) + f(x - u) - 2f(x)] \frac{\sin nu}{u} du \right| < \varepsilon.$$

But by virtue of Note 1 of the lemma in § 31

$$\int_{\eta}^{\delta}\left[\frac{f(x+u)+f(x-u)-2f(x)}{u}\right]\sin nu\,du \to 0 \quad \text{as} \quad n\to\infty.$$

From this, it follows that

$$\lim_{n\to\infty}\int_{0}^{\delta}[f(x+u)+f(x-u)-2f(x)]\frac{\sin nu}{u}\,du=0$$

and from (37.9) it follows that the convergence is proved.

In particular, if it is supposed that

$$\varphi_x(u)=f(x+u)+f(x-u)-2f(x).$$

then Dini's test gives: if the function $f(x)$ is continuous at the point x and

$$\int_{0}^{\delta}\frac{|\varphi_x(t)|}{t}\,dt \tag{38.1}$$

has meaning, then $\sigma(f)$ converges to $f(x)$ at the point x.

A number of corollaries can be derived from this. For example, if $f(x)$ in the neighbourhood of the point x satisfies the Lipschitz condition of order $\alpha > 0$, i.e. if

$$|f(x+u)-f(x)| \leqslant K|u|^{\alpha}$$

for $|u| \leqslant \delta$, then the integral (38.1) has meaning, which signifies that $\sigma(f)$ converges to $f(x)$. If the function $f(x)$ has a finite derivative at the point x, then in the neighbourhood of this point it satisfies the Lipschitz condition of order $\alpha = 1$ and therefore:

At the point x, where $f(x)$ possesses a finite derivative, its Fourier series converges to it.

In particular, *if $f(x)$ is differentiable everywhere in $(-\pi, \pi)$, then its Fourier series converges everywhere in this interval.*

§ 39. Jordan's test

As is known, any function of bounded variation is the difference of two non-decreasing bounded functions. If the function is monotonic, then it only has a discontinuity of the first kind. Moreover, if a function of bounded variation is continuous, then it is possible to represent it as the difference of two continuous non-decreasing functions.

We use the facts to prove the following theorem:

JORDAN'S THEOREM. *If $f(x)$ is of bounded variation in some interval (a, b), then its Fourier series converges at every point of this interval. Its sum is $f(x)$ at a point of continuity and $[f(x+0)+f(x-0)]/2$ at a point of discontinuity. Finally, if (a', b')*

lies entirely inside the interval (a, b), *where* $f(x)$ *is continuous, then the Fourier series converges uniformly in* (a', b').

From remarks made earlier, it is clear that it is sufficient to prove the theorem for the case of non-decreasing $f(x)$. In this case, supposing

$$S = \frac{f(x+0) + f(x-0)}{2},$$

we see from

$$f(x+u) + f(x-u) - 2S = [f(x+u) - f(x+0)] + [f(x-u) - f(x-0)],$$

that at a fixed x, each function in brackets is a monotonic function of u. We will now estimate

$$\int_0^\delta [f(x+u) - f(x+0)] \frac{\sin nu}{u} du. \tag{39.1}$$

Here δ is chosen so that $x \pm \delta \in (a, b)$. But whatever $\varepsilon > 0$, it is possible to take $\delta_1 < \delta$ so small that

$$|f(x+u) - f(x+0)| < \varepsilon \quad 0 \leqslant u \leqslant \delta_1.$$

Since $f(x+u) - f(x+0)$ does not decrease and is non-negative, then applying the second mean value theorem we see that

$$\int_0^{\delta_1} [f(x+u) - f(x+0)] \frac{\sin nu}{u} du$$

$$= [f(x+\delta_1) - f(x+0)] \int_{\delta_2}^{\delta_1} \frac{\sin nu}{u} du, \tag{39.2}$$

where $0 < \delta_2 < \delta_1$. But since (see (35.7))

$$\left| \int_{\delta_2}^{\delta_1} \frac{\sin nu}{u} du \right| < \pi$$

for any positive δ_1 and δ_2, then the modulus of integral (39.2) does not exceed $\pi\varepsilon$.

On the basis of the lemma of § 31

$$\left| \int_0^\delta [f(x+u) - f(x+0)] \frac{\sin nu}{u} du \right| < 2\pi\varepsilon,$$

if n is sufficiently large.

In the same way,

$$\int_0^\delta [f(x-u) - f(x-0)] \frac{\sin nu}{u} du.$$

is estimated.

Therefore, for sufficiently large n

$$\left|\int_0^\delta [f(x+u)+f(x-u)-2S]\frac{\sin nu}{u}\,du\right| \leqslant 4\pi\varepsilon.$$

where ε is as small as desired, and then on the basis of the convergence test of § 37 we see that the series converges at the point x to the value S.

Now let $f(x)$ be continuous in some interval $[a, b]$, and $[a', b']$ be any interval lying completely within (a, b).

It is possible to choose δ_1 so small that

$$|f(x+u)-f(x)| < \varepsilon \quad \text{and} \quad |f(x-u)-f(x)| < \varepsilon,$$

if $a' \leqslant x \leqslant b'$ and $0 \leqslant u \leqslant \delta_1$. If this is so, then in the preceding estimates of the integrals, x can be taken anywhere in (a', b') and therefore

$$\left|\int_0^\delta [f(x+u)+f(x-u)-2f(x)]\frac{\sin nu}{u}\,du\right| \leqslant 4\pi\varepsilon$$

for $a' \leqslant x \leqslant b'$, because of the test in § 37, which means that the series converges uniformly in (a', b').

Jordan's theorem has been proved.

From the given theorem, it follows in particular that *if $f(x)$ is of bounded variation in the whole interval $[-\pi, \pi]$ and continuous in it, whilst $f(-\pi) = f(\pi)$, then its Fourier series converges uniformly in $-\infty < x < +\infty$.*

Therefore: *the Fourier series for any periodic absolutely continuous function converges uniformly to it in $-\infty < x < +\infty$.*

Note. An important particular case of the given theorem was considered by Dirichlet. He investigated the case when the function $f(x)$ is bounded and has only a finite number of maxima and minima and no more than a finite number of points of discontinuity. For these functions he proved the convergence of the Fourier series at every point. It is clear that these functions are all of bounded variation.

§ 40. Integration of Fourier series

Let $f(x)$ be summable and

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

Let us denote the primitive of $f(x)$ by $F(x)$. Then

$$F(x) = \frac{a_0}{2}x + C + \sum \frac{-b_n \cos nx + a_n \sin nx}{n}, \tag{40.1}$$

whilst the series on the right-hand side converges uniformly.

This theorem is due to Lebesgue. In order to prove it, it is sufficient to note that $F(x) - (a_0/2)x$ is the primitive of $f(x) - a_0/2$, it is absolutely continuous and has a period 2π (see § 23, No. 8).

Therefore, the Fourier series for $F(x) - (a_0/2)x$ converges uniformly to it. But it has the form (see § 23, No. 8)

$$\sum \frac{-b_n \cos nx + a_n \sin nx}{n}.$$

This concludes the proof.

As a corollary, we obtain for any a and b

$$\int_a^b f(x)\,dx = \frac{a_0 x}{2}\Big|_a^b + \sum_{n=1}^{\infty} \frac{-b_n \cos nx + a_n \sin nx}{n}\Big|_a^b,$$

i.e. *Fourier series (even divergent) can be integrated term by term in any interval.*

Corollary. In formula (40.1) the series converges for all x; in particular, at $x = 0$; but this indicates the convergence of the series

$$\sum_{n=1}^{\infty} \frac{b_n}{n}.$$

Thus: *for any Fourier–Lebesgue series, the series* $\sum_{n=1}^{\infty} b_n/n$ *converges.*

This theorem makes it possible in some cases to establish immediately that the given series is not a Fourier–Lebesgue series. Thus, for example, the series

$$\sum_{n=2}^{\infty} \frac{\sin nx}{\ln n}$$

is not a Fourier–Lebesgue series, although from Theorem 1 § 30 it converges at every point.

On the other hand, the series $\sum_{n=1}^{\infty} a_n/n$ can also diverge; in particular, the series

$$\sum_{n=2}^{\infty} \frac{\cos nx}{\ln n},$$

for which the series $\sum a_n/n = \sum 1/(n \ln n)$ diverges, is nevertheless a Fourier–Lebesgue series (this was proved in § 30).

§ 41. Gibbs's phenomenon

We have proved in § 39 that for a function of bounded variation the Fourier series converges at every point, particularly, at points of discontinuity. We want to study in more detail the behaviour of the partial sums of the series $\sigma(f)$ at those points where $f(x)$ is discontinuous. Let us start with an investigation of a special case and then transfer to a general case.

Let $f(x) = x$ in $(-\pi, \pi)$ and $f(x)$ have a period 2π. Since $f(x)$ is odd, then its Fourier series consists of sines only and (see § 8)

$$b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx \, dx = \frac{2}{\pi} \int_0^\pi x \sin nx \, dx.$$

Integrating by parts, we find that

$$b_n = \frac{2}{\pi} \cdot \frac{-x \cos nx}{n} \Bigg|_0^\pi + \frac{2}{n\pi} \int_0^\pi \cos nx \, dx = 2(-1)^{n-1} \frac{1}{n}.$$

Thus

$$f(x) \sim 2 \left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots + (-1)^{n-1} \frac{\sin nx}{n} + \cdots \right].$$

Since $f(x)$ is of bounded variation, then its Fourier series converges everywhere to $f(x)$ at its points of continuity and to $[f(x+0) + f(x-0)]/2$ at the points of discontinuity of the first kind. Therefore we have for $x \neq \pm \pi$

$$x = 2 \left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots \pm \frac{\sin nx}{n} \mp \cdots \right],$$

if $x = \pm\pi$, then the series converges to 0 (which is evident immediately, as all its terms then equal zero).

If we make the change in variable $x = \pi - t$, then when x passes through the interval $[-\pi, \pi]$, the variable t will pass through the interval $[0, 2\pi]$, whence it follows that

$$\frac{\pi - t}{2} = \frac{\sin(\pi - t)}{1} - \frac{\sin 2(\pi - t)}{2} + \frac{\sin 3(\pi - t)}{3} - \cdots \pm \frac{\sin n(\pi - t)}{n} \pm \cdots$$

$$= \frac{\sin t}{1} + \frac{\sin 2t}{2} + \frac{\sin 3t}{3} + \cdots + \frac{\sin nt}{n} + \cdots, \tag{41.1}$$

if $t \neq 0$ and $t \neq 2\pi$. At these points the series on the right-hand side of (41.1) converges to zero.

We have already said in § 30 that this series will play an important role in many problems of the theory of trigonometric series. In § 30 it was proved the partial sums of series (41.1) are all bounded, i.e. there exists a constant C such that

$$\left| \sum_{k=1}^{n} \frac{\sin kx}{k} \right| \leqslant C, \quad \begin{array}{l} -\infty < x < +\infty, \\ n = 1, 2, \ldots. \end{array}$$

However, later it will be necessary for us to study in more detail the behaviour of these partial sums in the neighbourhood of the point $x = 0$.

We have

$$S_n(x) = \sum_{k=1}^{n} \frac{\sin kx}{k} = \int_0^x \left(\sum_{k=1}^{n} \cos kt \right) dt = \int_0^x \left[D_n(t) - \frac{1}{2} \right] dt,$$

where, as always, $D_n(t)$ is a Dirichlet kernel. Therefore

$$S_n(x) = \int_0^x D_n(t)\,dt - \frac{x}{2}. \tag{41.2}$$

But we know that (see (32.3))

$$D_n(t) = \frac{\sin nt}{t} + g(t)\sin nt + \frac{1}{2}\cos nt,$$

where $g(t)$ is bounded.

Supposing

$$\psi_x(t) = \begin{cases} g(t) & \text{for } 0 \leqslant t \leqslant x, \\ 0 & \text{for } x < t \leqslant 2\pi, \end{cases}$$

$$\psi_x(t + 2\pi) = \psi_x(t)$$

and using Note 3 of the lemma in § 31, we conclude that

$$\int_0^x D_n(t)\,dt = \int_0^x \frac{\sin nt}{t}\,dt + o(1) \tag{41.3}$$

uniformly in $0 \leqslant x \leqslant 2\pi$; therefore from (41.2) and (41.3)

$$S_n(x) = -\frac{x}{2} + \int_0^x \frac{\sin nt}{t}\,dt + o(1)$$

or

$$S_n(x) = -\frac{x}{2} + \int_0^{nx} \frac{\sin t}{t}\,dt + o(1). \tag{41.4}$$

If

$$\psi(x) = \frac{\pi - x}{2} \quad \text{in} \quad 0 < x < 2\pi, \tag{41.5}$$

$$\psi(x + 2\pi) = \psi(x),$$

then the function $\psi(x)$ has the form given in Fig. 6. We have already seen that the series (41.1) is $\sigma(\psi)$ and it converges everywhere to $\psi(x)$, apart from the points $x = 0$ and $x = 2\pi$, where it converges to zero.

Allowing x to take the values

$$x = \frac{\pi}{n}, \quad x = \frac{2\pi}{n}, \quad \ldots, \quad x = \pi,$$

we see from (41.4) that

$$\left.\begin{aligned} S_n\left(\frac{\pi}{n}\right) + \frac{\pi}{2n} &= \int_0^{\pi} \frac{\sin t}{t}\,dt + o(1), \\ S_n\left(\frac{2\pi}{n}\right) + \frac{\pi}{n} &= \int_0^{2\pi} \frac{\sin t}{t}\,dt + o(1), \\ &\cdots\cdots\cdots\cdots \\ S_n\left(\frac{k\pi}{n}\right) + \frac{k\pi}{2n} &= \int_0^{k\pi} \frac{\sin t}{t}\,dt + o(1). \end{aligned}\right\} \tag{41.6}$$

Taking into account what was said in § 35 concerning the behaviour of the curve $y = (\sin x)/x$, it is immediately seen that the curves $y = S_n(x)$ pass through the origin

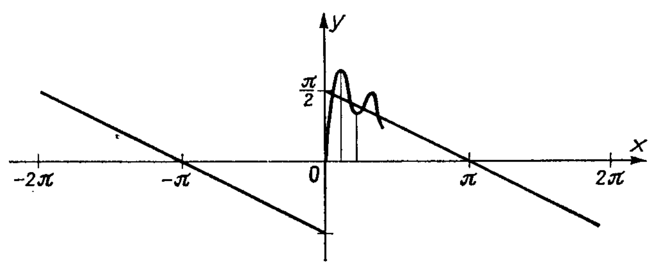

FIG. 6

of the co-ordinates, fluctuate around the straight line $y = \psi(x)$ and although for any x, $0 < x < \pi$, we have

$$\lim_{n\to\infty} S_n(x) = \psi(x),$$

however, from (41.6) it is evident that the curves $y = S_n(x)$ to the right of the point $x = 0$ concentrate round the interval $(0, l)$ where

$$l = \int_0^{\pi} \frac{\sin t}{t}\,dt.$$

This type of picture is also obtained on the left of $x = 0$, since all $S_n(x)$ are odd functions. Therefore, around the point $x = 0$ the curves oscillate not between $-\pi/2$ and $\pi/2$, as would be expected, but are concentrated round the interval $[-l, l]$. But calculation shows that $l = 1{\cdot}8519\ldots$, and since $\pi/2 = 1{\cdot}57\ldots$, then the length of the interval $[-l, l]$ exceeds the length of $[-\pi/2, \pi/2]$.

This circumstance was first noticed by Gibbs[11], which is why it is known as *Gibbs's phenomenon*, and the ratio l to $\pi/2$ is *Gibbs's constant*; this constant equals $1{\cdot}17$.

We will show that Gibbs's phenomenon holds for any function of bounded variation about its points of discontinuity, as long as they are isolated. Indeed, in a function of bounded variation the points of discontinuity are only of the first kind.

Let $f(x)$ be such a function and x_0 be an isolated point of discontinuity. If $f(x_0 + 0) - f(x_0 - 0) = d$, then the function

$$\varrho(x) = f(x) - \frac{d}{\pi}\psi(x - x_0)$$

is continuous in a sufficiently small neighbourhood of the point x_0, since $\varrho(x_0 \pm 0) = f(x_0 \pm 0) - (d/\pi)\psi(\pm 0)$ and therefore

$$\varrho(x_0 + 0) - \varrho(x_0 - 0) = d - \frac{d}{\pi}[\psi(+0) - \psi(-0)] = 0.$$

Since there are no other points of discontinuity for $f(x)$ in the neighbourhood being considered, if this neighbourhood were chosen to be sufficiently small, then $\varrho(x)$ is continuous in this neighbourhood and is of bounded variation in $[0, 2\pi]$. This means that its Fourier series converges uniformly in a sufficiently small neighbourhood of x_0; therefore the behaviour of the partial sums of the Fourier series for $f(x)$ around x_0 will be just the same as for $(d/\pi)\psi(x - x_0)$, i.e. as for $(d/\pi)\psi(x)$ around $x = 0$; therefore, Gibbs's phenomenon should also occur here.

From Riemann's principle of localization (see § 33) this is true if $f(x) \in L[-\pi, \pi]$ is of bounded variation in $[a, b]$ and x_0 is an isolated point of discontinuity of $f(x)$ in $[a, b]$.

§ 42. Determination of the magnitude of the discontinuity of a function from its Fourier series

Let us assume that at some point x the function $f(x)$ has a discontinuity of the first kind, whilst

$$f(x + 0) - f(x - 0) = d. \tag{42.1}$$

The magnitude of this discontinuity can be determined from the following formula (see Lukács[11]):

$$\lim_{n\to\infty} \frac{\overline{S}_n(x)}{\ln n} = -\frac{d}{\pi}. \tag{42.2}$$

In fact, we have

$$f(x + t) - f(x - t) = d + \varepsilon(t), \quad \text{where} \quad \varepsilon(t) \to 0 \quad \text{as} \quad t \to 0.$$

But from formula (31.9) due to the oddness of $\overline{D}_n(t)$ we have

$$\overline{S}_n(x) = -\frac{1}{\pi}\int_0^\pi [f(x + t) - f(x - t)]\,\overline{D}_n(t)\,dt,$$

therefore

$$\overline{S}_n(x) = -\frac{d}{\pi}\int_0^\pi \overline{D}_n(t)\,dt - \frac{1}{\pi}\int_0^\pi \varepsilon(t)\overline{D}_n(t)\,dt. \tag{42.3}$$

We will prove first that

$$\lim_{n\to\infty}\frac{1}{\ln n}\int_0^\pi \overline{D}_n(t)dt = 1. \tag{42.4}$$

Indeed, supposing that $\nu = (n-1)/2$, we have

$$\int_0^\pi \overline{D}_n(t)\,dt = -\sum_{k=1}^{n}\frac{\cos kt}{k}\Big|_0^\pi = 2\left(1+\frac{1}{3}+\cdots+\frac{1}{2\nu+1}\right)$$

$$= 2\left[\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2\nu}+\frac{1}{2\nu+1}\right)-\frac{1}{2}\left(1+\frac{1}{2}+\cdots+\frac{1}{\nu}\right)\right]$$

$$\approx 2\left[\ln\nu - \frac{1}{2}\ln\nu\right] = \ln\nu \approx \ln n. \tag{42.5}$$

Thus, formula (42.4) is proved.
We will prove now that

$$\lim_{n\to\infty}\frac{1}{\ln n}\int_0^\pi \varepsilon(t)\overline{D}_n(t)\,dt = 0. \tag{42.6}$$

For this we will take any $\eta > 0$ and choose δ such that

$$|\varepsilon(t)| < \eta \quad \text{at} \quad 0 \leqslant t \leqslant \delta.$$

Then

$$\left|\int_0^\delta \varepsilon(t)\overline{D}_n(t)\,dt\right| < \eta\int_0^\delta \left|\overline{D}_n(t)\right|dt < C\eta\ln n \tag{42.7}$$

(from (35.18)) where C is a constant. Moreover, since

$$\left|\overline{D}_n(t)\right| \leqslant \frac{\pi}{\delta} \quad \text{at} \quad \delta \leqslant t \leqslant \pi \tag{42.8}$$

from (29.11) it follows that

$$\int_\delta^\pi \varepsilon(t)\overline{D}_n(t)\,dt = O(1),$$

and therefore, (42.6) follows from (42.7) and (42.8). From (42.3), (42.4) and (42.6), the truth of formula (42.2) now follows.

COROLLARY 1. *At any point of discontinuity of the first kind, the series conjugate to the Fourier series for $f(x)$ diverges.*

Indeed, at this point

$$\bar{S}_n(x,f) = -\frac{d}{\pi}\ln n + \varepsilon_n \ln n,$$

where $\varepsilon_n \to 0$.

COROLLARY 2. *If $f(x)$ is continuous at the point x, then $\bar{S}_n(f, x) = o(\ln n)$; if $\bar{S}_n(x,f) = o(\ln n)$, then the point x cannot be a point of discontinuity of the first kind.*

COROLLARY 3. *If for the function $f(x)$, the Fourier coefficients are of order $o(1/n)$, then there can be no points of discontinuity of the first kind in it.*

Indeed, then

$$\bar{S}_n(f, x) = o\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) = o(\ln n)$$

(see Introductory Material, § 11).

From this we conclude in particular that:

If $f(x)$ is of bounded variation and the Fourier coefficients are of order $o(1/n)$, then it is continuous.

Indeed, in a function $f(x)$ of bounded variation, points of discontinuity can only be of the first kind; but from Corollary 3 it follows that such points cannot exist and therefore $f(x)$ is continuous.

However, it must be stated that if $f(x)$ is continuous and of bounded variation, then its Fourier coefficients are of order $o(1/n)$. We will prove this in Chapter II, § 2.

§ 43. Singularities of Fourier series of continuous functions. Fejér polynomials

We want to show that if no limitations are imposed on the function $f(x)$ except continuity, then its Fourier series can also diverge at some point and converge non-uniformly about some point, although it converges everywhere. The first examples of this kind were given by du Bois-Reymond[1] and Lebesgue, therefore it is customary to refer to these facts as *du Bois-Reymond's singularity* (for the case of divergence) and *Lebesgue's singularity* (for the case of non-uniform convergence).

Here, following Fejér[2], we will establish some trigonometric polynomials, from which functions will be constructed possessing either one or other of these singularities. Subsequently (in Chapter IV) these Fejér polynomials will help in the construction of considerably more complicated examples, namely: continuous functions, in which the Fourier series diverges in an everywhere dense set, or in a set of the power of the continuum and also continuous functions in which the series converges everywhere but non-uniformly in any interval δ, lying in $[-\pi, \pi]$.

Constructional elements. Let us consider two trigonometric polynomials

$$Q(x, n) = \frac{\cos nx}{n} + \frac{\cos(n+1)x}{n-1} + \cdots + \frac{\cos(2n-1)x}{1}$$
$$- \left[\frac{\cos(2n+1)x}{1} + \frac{\cos(2n+2)x}{2} + \cdots + \frac{\cos 3nx}{n}\right], \tag{43.1}$$

$$\bar{Q}(x, n) = \frac{\sin nx}{n} + \frac{\sin(n+1)x}{n-1} + \cdots + \frac{\sin(2n-1)x}{1}$$
$$- \left[\frac{\sin(2n+1)x}{1} + \frac{\sin(2n+2)x}{2} + \cdots + \frac{\sin 3nx}{n}\right]. \tag{43.2}$$

Let us note their properties as follows:

(a) There exists a constant C such that

$$\left|Q(x,\, n)\right| \leqslant C \quad \text{and} \quad \left|\bar{Q}(x,\, n)\right| \leqslant C \tag{43.3}$$

for any x and n.

In fact

$$Q(x,\, n) = \sum_{k=1}^{n} \frac{\cos(2n-k)x - \cos(2n+k)x}{k} = 2\sin 2nx \sum_{k=1}^{n} \frac{\sin kx}{k},$$

$$\bar{Q}(x,\, n) = \sum_{k=1}^{n} \frac{\sin(2n-k)x - \sin(2n+k)x}{k} = -\,2\cos 2nx \sum_{k=1}^{n} \frac{\sin kx}{k}.$$

But, as is known (see (30.8)) we have

$$\left|\sum_{k=1}^{n} \frac{\sin kx}{k}\right| \leqslant M \quad (-\infty < x < +\infty,\ n = 1, 2, \ldots).$$

Therefore, supposing $C = 2M$, we see that property (a) is proved.

(b) If we denote by $\varphi(x, Q)$ or $\varphi(x, \bar{Q})$ any partial sum of the polynomial $Q(x)$ or $\bar{Q}(x)$, (i.e. the sum of any number of the first terms in the polynomial), then

$$\left.\begin{aligned} |\varphi(x,\, Q)| &\leqslant 2(1 + \ln n) \\ \text{and}\quad \left|\varphi(x,\, \bar{Q})\right| &\leqslant 2(1 + \ln n), \end{aligned}\right\} \tag{43.4}$$

because

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} < 1 + \ln n.$$

(c) If $\delta \leqslant x \leqslant \pi$, then

$$\left|\varphi(x,\, Q)\right| \leqslant M_\delta \quad \text{and} \quad \left|\varphi(x,\, \bar{Q})\right| \leqslant M_\delta, \tag{43.5}$$

where M_δ is a constant depending only on δ.

Indeed, every sum $\varphi(x, Q)$ has either the form

$$\sum_{k=0}^{p} \frac{\cos(n+k)x}{n-k} \quad \text{for} \quad p \leqslant n-1,$$

or the form

$$\sum_{k=0}^{n-1} \frac{\cos(n+k)x}{n-k} - \sum_{k=1}^{p} \frac{\cos(2n+k)x}{k} \quad \text{for} \quad p \leqslant n.$$

This means that each of the sums in the expression $\varphi(x, Q)$ has the form $\sum \alpha_k \cos(n+k)\,x$, where the numbers α_k are positive, decrease or increase monotonically and do not exceed 1; therefore, using the corollary of Abel's transformation (see Introductory Material, § 1), we see that each such sum does not exceed the constants depending only on δ. The same argument holds for $\varphi(x, \bar{Q})$, since there everything is the same except that sines are substituted for cosines.

(d) Finally, we assume

$$P(x, n) = \frac{\cos nx}{n} + \frac{\cos(n+1)x}{n-1} + \cdots + \frac{\cos(2n-1)x}{1}, \tag{43.6}$$

$$\bar{P}(x, n) = \frac{\sin nx}{n} + \frac{\sin(n+1)x}{n-1} + \cdots + \frac{\sin(2n-1)x}{1}, \tag{43.7}$$

i.e. $P(x, n)$ is the sum of the first n terms of $Q(x, n)$ and $\bar{P}(x, n)$ is the sum of the first n terms of $\bar{Q}(x, n)$. Then we have

$$P(0, n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} > \ln n, \tag{43.8}$$

$$\bar{P}\left(\frac{\pi}{4n}, n\right) = \frac{\sin n \dfrac{\pi}{4n}}{n} + \cdots + \frac{\sin(2n-1)\dfrac{\pi}{4n}}{1}$$

$$> \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) \sin\frac{\pi}{4} > \frac{1}{\sqrt{2}} \ln n,$$

Therefore,

$$\bar{P}\left(\frac{\pi}{4n}, n\right) > \frac{\ln n}{\sqrt{2}}. \tag{43.9}$$

We will use these facts for establishing the examples required.

§ 44. A continuous function with a Fourier series which converges everywhere but not uniformly

Let $a > 1$ be an integer, which we will select later. Let us suppose that

$$n_k = a^{k^2} \tag{44.1}$$

and define

$$\bar{Q}_k(x) = \bar{Q}(x, n_k), \tag{44.2}$$

where $\bar{Q}(x, n)$ is a trigonometric polynomial, defined by formula (43.2).

Let us assume

$$g(x) = \sum_{k=2}^{\infty} \frac{1}{k^2} \bar{Q}_k(x) \tag{44.3}$$

and prove if a is chosen suitably that $g(x)$ is a function with the properties given in the title of this section.

Indeed, from (43.3) for all x and k

$$|\bar{Q}_k(x)| < C, \tag{44.4}$$

and therefore series (44.3) converges absolutely and uniformly, which means $g(x)$ is continuous. Since for any $a > 1$ and for $k \geqslant 2$ we have

$$a^{k^2} > 3a^{(k-1)^2},$$

i.e. (see (44.1))

$$n_k > 3n_{k-1},$$

then from (44.2) it follows that no term containing $\sin nx$ appears simultaneously for any n in two different $\bar{Q}_k(x)$, therefore in the series (44.3) all the sines, as in a normal trigonometric series, are arranged in ascending order of the multiplier n for x.

On the basis of the lemma of § 12, series (44.3) is the Fourier series of $g(x)$, because its partial sums with indices $3n_k$ converge uniformly to $g(x)$.

We will prove that the partial sums $S_n(x, g)$ of the Fourier series for $g(x)$ are all bounded.

Indeed, each such sum has the form

$$S_n(x, g) = \sum_{k=1}^{m} \frac{1}{k^2} \bar{Q}_k(x) + \frac{1}{(m+1)^2} \varphi(x, \bar{Q}_{m+1}) \tag{44.5}$$

(in particular cases the second term of the sum (44.5) can disappear). But then on the basis of (44.2) and (43.3) we have

$$\left| \sum_{k=2}^{m} \frac{1}{k^2} \bar{Q}_k(x) \right| \leqslant C \sum_{k=2}^{m} \frac{1}{k^2} < A, \tag{44.6}$$

where A is an absolute constant. Moreover, on the basis of (44.2), (43.2) and (43.4) we have

$$\left| \frac{1}{(m+1)^2} \varphi(x, \bar{Q}_{m+1}) \right| \leqslant \frac{1}{(m+1)^2} 2\,(1 + \ln a^{(m+1)^2}) < 2(1 + \ln a) \tag{44.7}$$

and, therefore, from (44.5), (44.6) and (44.7)

$$|S_n(x, g)| \leqslant B \quad (n = 0, 1, \ldots; \ 0 \leqslant x \leqslant 2\pi), \tag{44.8}$$

where B is an absolute constant.

We will note in passing (this will be necessary in Chapter IV) that, supposing

$$S_n(x, g) - g(x) = R_n(x, g),$$

we have

$$|R_n(x, g)| \leqslant K \quad (n = 1, 2, \ldots; \ -\pi \leqslant x \leqslant \pi), \tag{44.9}$$

where K is an absolute constant, which follows from the fact that $g(x)$ is bounded and from (44.8).

Let us turn to a study of the convergence of series $\sigma(g)$.

We will first remark that for any $\delta > 0$ in the interval $\delta \leqslant x \leqslant \pi$ (which also means $-\pi \leqslant x \leqslant -\delta$) the Fourier series for $g(x)$ converges uniformly.

Indeed, from formula (44.5) it is evident that

$$R_n(x, g) = \frac{1}{(m+1)^2} \varphi(x, \bar{Q}_{m+1}) - \sum_{m+1}^{\infty} \frac{1}{k^2} \bar{Q}_k(x),$$

and then from (43.3) and (43.5) it follows that

$$|R_n(x, g)| \leqslant C \sum_{m+1}^{\infty} \frac{1}{k^2} + \frac{M_\delta}{(m+1)^2} < \varepsilon,$$

if n and therefore m is sufficiently large.

Thus we see that the Fourier series for $g(x)$ converges for any $x \not\equiv 0 \pmod{2\pi}$. But for $x \equiv 0 \pmod{2\pi}$ it should also converge, since it consists only of sines.

It remains for us to prove that the series $\sigma(g)$ converges non-uniformly near $x = 0$.

For this purpose we consider its partial sums with indices

$$\nu_m = 2n_m - 1.$$

Every such sum has the form

$$S_{\nu_m}(x) = \sum_{k=1}^{m-1} \frac{1}{k^2} \bar{Q}_k(x) + \frac{1}{m^2} \bar{P}(x, n_m),$$

therefore

$$R_{\nu_m}(x) = S_{\nu_m}(x) - g(x) = \frac{1}{m^2} \bar{P}(x, n_m) - \sum_{m}^{\infty} \frac{1}{k^2} \bar{Q}_k(x).$$

Supposing

$$x_m = \frac{\pi}{4n_m},$$

we find from (43.3) and (43.9)

$$R_{\nu_m}(x_m) > \frac{1}{m^2} \bar{P}\left(\frac{\pi}{4n_m}, n_m\right) - C\sum_{m}^{\infty} \frac{1}{k^2} > \frac{1}{m^2} \frac{1}{\sqrt{2}} \ln a^{m^2} - \frac{C}{m} = \frac{\ln a}{\sqrt{2}} - C > 1,$$

provided we choose a so that

$$\ln a > \sqrt{2}\,(1 + C).$$

Thus

$$R_{\nu_m}(x_m) > 1 \quad (m = 1, 2, \ldots) \tag{44.10}$$

for some sequence of points x_m tending to 0, which means that the Fourier series for $g(x)$ converges non-uniformly near $x = 0$.

The theorem is proved.

§ 45. Continuous function with a Fourier series divergent at one point (Fejér's example)

We shall consider Fejér's polynomials $Q(x, n)$, described in § 43, and by using them establish the Fourier series of continuous functions divergent at $x = 0$; in this case, series will be obtained as desired possessing either bounded or unbounded partial sums. These and other examples will be used later (in Chapter IV) for constructions of more complex character.

We assume first as in the preceding section

$$n_k = a^{k^2},$$

where a is an integer and $a \geqslant 2$; let us suppose

$$Q_k(x) = Q(x, n_k), \tag{45.1}$$

and let

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k^2} Q_k(x). \tag{45.2}$$

We will again see as in the preceding section that $f(x)$ is continuous and the series (45.2), if every term of any polynomial $Q_k(x)$ in it is considered separately (but not grouped in sums), is its Fourier series.

We see, just as in the proof of (44.8), that

$$|S_n(x, f)| \leqslant B \tag{45.3}$$

for any n and x and that the series $\sigma(f)$ converges uniformly in $(-\pi \leqslant x \leqslant -\delta)$ and $(\delta \leqslant x \leqslant \pi)$, i.e. it converges for any $x \not\equiv 0 \pmod{2\pi}$. But at $x = 0$ it diverges, since supposing that

$$\nu_m = 2n_m - 1, \quad \mu_m = 3n_{m-1},$$

we have

$$S_{\nu_m}(0) - S_{\mu_m}(0) = \frac{P(0, n_m)}{m^2} > \frac{\ln n_m}{m^2} = \frac{m^2 \ln a}{m^2} = \ln a > 0, \quad m = 1, 2, \ldots$$

Consequently, Cauchy's test of convergence is not fulfilled.

Thus, $\sigma(f)$ diverges at $x = 0$, although its partial sums are all bounded by virtue of (45.3).

If instead of $n_k = a^{k^2}$ we supposed

$$n_k = a^{k^3} \quad (a > 2),$$

then we would obtain

$$S_{\nu_m}(0) - S_{\mu_m}(0) = \frac{P(0, n_m)}{m^2} > \frac{m^3 \ln a}{m^2} = m \ln a,$$

i.e. the series would not only diverge at the point $x = 0$, but would have unbounded partial sums at this point.

§ 46. Divergence at one point (Lebesgue's example)

The preceding examples of Fejér (see § 45) although suitable for use in further constructions possess one disadvantage; since the corresponding functions were established purely analytically with the help of formulae, it is not possible to represent them by curves and understand geometrically why the divergence of the Fourier series occurs.

Therefore, we will describe Lebesgue's example (only slightly modified in order to shorten the proof), where it is possible to represent the function graphically though only approximately.

Let $n_1, n_2, \ldots, n_k, \ldots$ be a sequence of integers which we shall define later. Let us suppose

$$a_0 = 1, \quad a_k = n_1 n_2 \ldots n_k \quad (k = 1, 2, \ldots).$$

and define

$$I_k = \left(\frac{\pi}{a_k}, \frac{\pi}{a_{k-1}}\right) \quad (k = 1, 2, \ldots).$$

We shall later define a sequence of numbers c_k, whilst now we only assume that $c_k \downarrow 0$.

Let

$$f(x) = c_k \sin a_k x \quad \text{in} \quad I_k,$$

$$f(0) = 0,$$

$$f(-x) = f(x).$$

It is clear that $f(x)$ is defined everywhere in $[-\pi, \pi]$, it is continuous in each I_k and reverts to 0 at its end points, i.e. it has no discontinuities at finite points; finally, $f(x) \to 0$ as $x \to 0$ (Fig. 7) since $c_k \downarrow 0$, which means that $f(x)$ is continuous everywhere.

We shall prove that its Fourier series converges everywhere in $[-\pi, \pi]$ apart from $x = 0$. Since $f(x)$ has only a finite number of maxima and minima in $[\delta, \pi]$ it is of

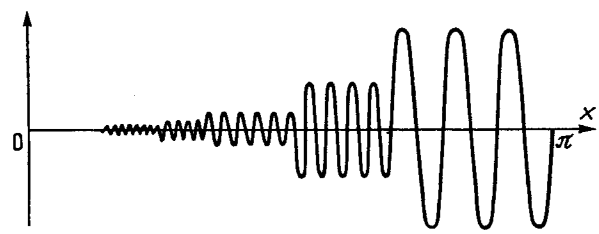

FIG. 7

bounded variation in this interval (and also in $[-\pi, -\delta]$). This means that its Fourier series converges at each point $[-\pi, \pi]$, apart from $x = 0$.

We will show that with a proper choice of the numbers c_k and n_k the series $\sigma(f)$ diverges at $x = 0$.

As is known, for any $f(x)$ we have

$$S_n(x, f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x + t) \frac{\sin nt}{t} dt + o(1),$$

which means that at $x = 0$

$$S_n(0, f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \frac{\sin nt}{t} dt + o(1).$$

Our $f(x)$ is even, therefore

$$S_n(0, f) = \frac{2}{\pi} \int_{0}^{\pi} f(t) \frac{\sin nt}{t} dt + o(1). \tag{46.1}$$

We will show that by a suitable choice of c_k and n_k, we have

$$J_k = \int_0^{\pi} f(t) \frac{\sin a_k t}{t} dt \to \infty \quad \text{as} \quad k \to \infty. \tag{46.2}$$

If this is so, then $S_{a_k}(0, f) \to +\infty$ as $k \to \infty$ (as is evident from (46.1)) and then the series $\sigma(f)$ diverges at $x = 0$.

In order to evaluate J_k, we will divide it into three terms

$$\begin{aligned} J_k &= \int_0^{\pi/a_k} f(t) \frac{\sin a_k t}{t} dt + \int_{\pi/a_k}^{\pi/a_{k-1}} f(t) \frac{\sin a_k t}{t} dt + \int_{\pi/a_{k-1}}^{\pi} f(t) \frac{\sin a_k t}{t} dt \\ &= J_k' + J_k'' + J_k'''. \end{aligned} \tag{46.3}$$

We have

$$\left| \frac{\sin a_k t}{t} \right| \leqslant a_k.$$

This means that

$$(J_k') \leqslant \max_{0 \leqslant t \leqslant \pi/a_k} |f(t)| \, a_k \frac{\pi}{a_k} = \pi c_{k+1} = o(1), \tag{46.4}$$

since $c_k \downarrow 0$.

Up to now we have not defined the numbers c_k and n_k. We will now suppose that $n_1 = 2$, $c_1 = 1$. If $c_1, c_2, \ldots, c_{k-1}$ and $n_1, n_2, \ldots, n_{k-1}$ are already defined, then $f(t)$ is defined in $I_1, I_2, \ldots, I_{k-1}$, i.e. in $(\pi/a_{k-1}, \pi)$. It is continuous in this semi-interval and $t \geqslant \pi/a_{k-1}$, therefore $[f(t)]/t$ is bounded. Consequently, if n is sufficiently large, then

$$\int_{\pi/a_{k-1}}^{\pi} \frac{f(t)}{t} \sin nt \, dt$$

can be made as small as desired (see § 19).

Since $a_k = n_1 n_2 \ldots n_k$, then if $n_1, \ldots, n_{k-1}$ are already fixed, n_k is still at our disposal, which means that by increasing it we can make a_k as large as desired, in particular such that

$$|J_k'''| \leqslant \left| \int_{\pi/a_{k-1}}^{\pi} f(t) \frac{\sin a_k t}{t} dt \right| < \frac{1}{k}, \tag{46.5}$$

whence it follows that $J_k''' = o(1)$ as $k \to \infty$.

It remains to estimate J_k''. We have

$$\begin{aligned} J_k'' &= \int_{\pi/a_k}^{\pi/a_{k-1}} c_k \sin a_k t \frac{\sin a_k t}{t} dt = \frac{c_k}{2} \int_{\pi/a_k}^{\pi/a_{k-1}} \frac{1 - \cos 2a_k t}{t} dt \\ &= \frac{c_k}{2} \ln n_k - \frac{c_k}{2} \int_{\pi/a_k}^{\pi/a_{k-1}} \frac{\cos 2a_k t}{t} dt. \end{aligned}$$

But according to the second mean value theorem, taking into account that $1/t$ is positive and decreases monotically in the range of integration, we find

$$\left|\int_{\pi/a_k}^{\pi/a_{k-1}} \frac{\cos 2a_k t}{t}\,dt\right| \leqslant \frac{a_k}{\pi}\left|\int_{\pi/a_k}^{\xi} \cos 2a_k t\,dt\right| \leqslant \frac{a_k}{\pi}\,\frac{2}{2a_k} = \frac{1}{\pi}.$$

Therefore from $c_k \to 0$ it follows that

$$J_k'' = \frac{1}{2} c_k \ln n_k + o(1),$$

whence from (46.4) and (46.5)

$$J_k = \frac{1}{2} c_k \ln n_k + o(1).$$

We can now assume, provided $c_k = 1/\sqrt{\ln n_k}$, that $c_k \downarrow 0$ and

$$J_k = \frac{1}{2}\sqrt{\ln n_k} + o(1) \quad \text{as} \quad k \to \infty.$$

Thus, the proof is concluded.

From Fig. 7 it is evident that the function $f(x)$, as x approaches zero, performs more and more frequent oscillations; thus, it is found graphically that the divergence of the series $\sigma(f)$ at $x = 0$ is produced by the fact that $f(x)$ is of unbounded variation in the neighbourhood of this point.

Note. Later (in Chapter V, § 22) we will need the example of a continuous function, for which the Fourier series converges to zero everywhere in $[0, 2\pi]$ external to some interval $[a, b]$, converges at every point of (a, b) and diverges either only at a or only at b or at both end points of the interval (a, b), and possesses unbounded partial sums at points of divergence (we say briefly: it diverges without bound). All such examples are easily obtained, following the method of establishing Lebesgue's example.

Indeed, if we suppose

$$\varphi(x) = \begin{cases} 0 & \text{in } [-\pi, 0], \\ f(x) & \text{in } [0, \pi], \end{cases}$$

then

$$S_n(0, \varphi) = \frac{1}{2} S_n(0, f),$$

and therefore $\sigma(\varphi)$ diverges without bound at $x = 0$; moreover, $\sigma(\varphi)$ converges in $0 < x \leqslant \pi$ and converges to zero in $(-\pi, 0)$, which follows from the principle of localization (see § 33). If we suppose

$$\varphi_a(x) = \varphi(x - a),$$

then we obtain a function for which $\sigma(\varphi_a)$ diverges at $x = a$, converges to zero in $[a - \pi, a]$ and converges in $(a, a + \pi]$.

The function $\Psi(x) = \varphi(-x)$ has a Fourier series which converges everywhere except at $x = 0$, where it diverges without bound, and moreover this series converges to zero at $0 < x \leqslant \pi$.

Therefore

$$\Psi_b(x) = \Psi(x - b)$$

has a series which converges everywhere except at $x = b$, where it diverges without bound, whilst it converges to zero in $(b, b + \pi)$.

Now let $0 < a < b < 2\pi$. Let us construct $\lambda(x)$ in the following way. We choose the points α and γ such that $0 < \alpha < a < \gamma < b$ and let

$$\lambda(x) = \begin{cases} 1 & \text{in} \quad (a, \gamma), \\ 0 & \text{outside} \quad (\alpha, b), \end{cases}$$

$\lambda(x)$ is interpolated linearly by (α, a) and (γ, b) (Fig. 8).

According to Steinhaus' theorem (§ 34) the series $\sigma(\lambda\varphi_a)$ is equiconvergent with $\lambda(x)\sigma(\varphi_a)$ and therefore it converges everywhere, apart from $x = a$, where it diverges without bound, whilst outside $[a, b)$ it converges everywhere to zero (either because $\lambda(x) = 0$ or because $\varphi_a(x) = 0$).

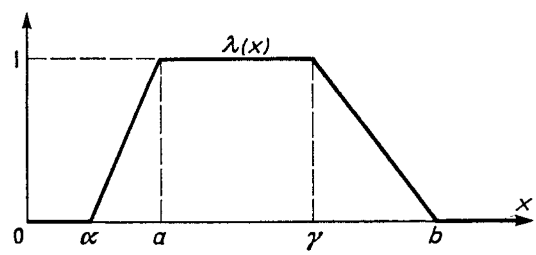

FIG. 8

In just the same way, if we denote by $\lambda^*(x)$ a function which is equal to 1 in (γ, b), to 0 outside (a, β) and can be interpolated linearly in (a, γ) and (b, β), where

$$0 < a < \gamma < b < \beta < 2\pi,$$

then we see that $\sigma(\lambda^*\Psi_b)$ is equiconvergent with $\lambda^*(x)\sigma(\Psi_b(x))$ and therefore it diverges without bound at $x = b$, converges everywhere apart from $x = b$ and converges to zero outside $(a, b]$.

Finally, supposing

$$F(x) = \lambda\varphi_a(x) + \lambda^*\Psi_b(x),$$

we see that $F(x)$ is continuous and $\sigma(f)$ diverges without bound at $x = a$ and $x = b$, but converges at all the other points, and moreover converges to zero everywhere outside $[a, b]$.

Note. The Fourier coefficients for those series which we established in §§ 45 and 46 tend to zero according to a rather complicated law. In connection with the solution of some problems in the theory of integral equations, the question arose: is it possible to find some continuous even function $f(x)$ for which

$$\sigma(f) = \sum_{n=1}^{\infty} a_n \cos nx, \quad \text{where} \quad |a_n| \downarrow 0$$

and moreover $\sigma(f)$ diverges at $x = 0$? Salem[14] gave an affirmative answer to this question. We will not give the proof here, since it is based on the study of some theoretical numerical inequalities, which would digress too far from the subject matter of this book.

§ 47. Summation of a Fourier series by Fejér's method

We have seen that even Fourier series of continuous functions have points of divergence (§§ 45 and 46). The question arises as to what extent the Fourier series can then be used for calculating the values of the function $f(x)$? Here, as is always the case with divergent series, it is natural to resort to one or other method of summation.

Let us recall (see Introductory Material, § 6) that the functional series is said to be *summable by the method* $(C, 1)$, if there exists a limit

$$\lim_{n\to\infty} \sigma_n(x),$$

where

$$\sigma_n(x) = \frac{1}{n+1} \sum_{k=0}^{n} S_k(x), \tag{47.1}$$

and $S_n(x)$ are the partial sums of the series.

The application of this method to Fourier series is usually known as *summation by Fejér's method,* since Fejér first drew attention to the usefulness of Cesàro sums in this case and proved the fundamental theorem. Later it was generalized by Lebesgue.

We know (see (31.3)) that the partial sum $S_n(x)$ of the Fourier series of the function $f(x)$ is expressed by the formula

$$S_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) D_n(t - x)\, dt,$$

where $D_n(u)$ is a Dirichlet kernel. Therefore a Cesàro sum, defined by (47.1), should have the form

$$\sigma_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \frac{1}{n+1} \sum_{k=0}^{n} D_k(t - x)\, dt = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) K_n(t - x)\, dt, \tag{47.2}$$

where

$$K_n(u) = \frac{1}{n+1} \sum_{k=0}^{n} D_k(u). \tag{47.3}$$

Consequently

$$\sigma_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x + u) K_n(u)\, du. \tag{47.4}$$

The function $K_n(u)$ is known as a *Fejér kernel*; we will now find an appropriate expression for it.

Since

$$D_n(u) = \frac{\sin\left(n + \dfrac{1}{2}\right)u}{2\sin\dfrac{u}{2}} = \frac{\cos nu - \cos(n+1)u}{4\sin^2\dfrac{u}{2}},$$

then

$$K_n(u) = \frac{1}{n+1}\sum_{k=0}^{n} \frac{\cos ku - \cos(k+1)u}{4\sin^2\dfrac{u}{2}} = \frac{1-\cos(n+1)u}{(n+1)4\sin^2\dfrac{u}{2}}$$

$$= \frac{1}{2(n+1)}\left(\frac{\sin(n+1)\dfrac{u}{2}}{\sin\dfrac{u}{2}}\right)^2.$$

Thus

$$K_n(u) = \frac{1}{2(n+1)}\left(\frac{\sin(n+1)\dfrac{u}{2}}{\sin\dfrac{u}{2}}\right)^2. \tag{47.5}$$

From this expression we immediately derive a number of properties of the kernel.

(1) $K_n(u) \geqslant 0$.

This property will play an essential role later.

(2) We have

$$K_n(u) \leqslant \frac{1}{2(n+1)\sin^2\dfrac{u}{2}} \leqslant \frac{\pi^2}{2(n+1)u^2} \quad \text{for} \quad 0 < |u| \leqslant \pi, \tag{47.6}$$

and therefore

$$K_n(u) = O\left(\frac{1}{nu^2}\right) \quad \text{for} \quad 0 < |u| \leqslant \pi \tag{47.7}$$

and

$$K_n(u) \leqslant \frac{\pi^2}{2(n+1)\delta^2} \quad \text{for} \quad 0 < \delta \leqslant |u| \leqslant \pi, \tag{47.8}$$

whence for any $\delta > 0$, supposing

$$M_n(\delta) = \max_{\delta \leqslant u \leqslant \pi} K_n(u),$$

we have

$$\lim_{n\to\infty} M_n(\delta) = 0. \tag{47.9}$$

(3) We have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} K_n(u)\,du = 1. \tag{47.10}$$

This follows from (47.3) and from

$$\frac{1}{\pi}\int_{-\pi}^{\pi} D_k(u)\,du = 1 \quad (k = 0, 1, 2, \ldots).$$

(4) If $\delta > 0$, then

$$\lim_{n\to\infty} \frac{1}{\pi}\int_{-\delta}^{\delta} K_n(u)\,du = 1. \tag{47.11}$$

This immediately follows from (47.9) and (47.10).

Starting from these properties, we can prove the following theorem, concerning the summation of Fourier series by Fejér's method.

FEJÉR'S THEOREM. *If x is a point of continuity of the function $f(x)$ or a point of discontinuity of the first kind, then at this point $\sigma(f)$ is summable by Fejér's method to $f(x)$ or to $[f(x+0)+f(x-0)]/2$, respectively; if (a, b) is an interval where $f(x)$ is continuous, then $\sigma(f)$ is uniformly summable by Fejer's method to $f(x)$ in any interval $[\alpha, \beta]$ lying within the interval (a, b).*

Finally, if $f(x)$ is everywhere continuous, then its Fourier series is uniformly summable by Fejér's method in $[-\pi, \pi]$, i.e. $\sigma_n(x)$ uniformly converges to $f(x)$ in this interval.

In order to prove this theorem we will turn to a lemma which is also useful in other circumstances.

LEMMA. *Let*

$$f_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\Psi_n(t)\,dt, \tag{47.12}$$

where the function $\Psi_n(t)$ possesses the following properties:

(1) *$\Psi_n(t)$ is an even function.*

(2) $\int_{-\pi}^{\pi} |\Psi_n(t)|\,dt \leqslant C\ (n = 1, 2, \ldots)$ *where C is a constant.*

(3) *Supposing for $\delta > 0$*

$$M_n(\delta) = \sup_{\delta\leqslant|t|\leqslant\pi} |\Psi_n(t)|,$$

we have

$$\lim_{n\to\infty} M_n(\delta) = 0;$$

(4) $\dfrac{1}{\pi}\int_{-\pi}^{\pi} \Psi_n(t)\,dt = 1.$

Then: if x is a point of discontinuity of the first kind for $f(x)$, then

$$f_n(x) \to \frac{f(x+0)+f(x-0)}{2} \quad \textit{as} \quad n\to\infty,$$

$f_n(x) \to f(x)$ at each point of continuity of $f(x)$.

If $f(x)$ is continuous in (a, b), then $f_n(x) \to f(x)$ uniformly in (α, β) for any $[\alpha, \beta] \subset (a, b)$.

To prove this lemma we will note first that from property (4) of the function $\Psi_n(t)$ we have

$$\frac{f(x+0)+f(x-0)}{2} = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f(x+0)+f(x-0)}{2}\Psi_n(t)\,dt$$

$$= \frac{1}{\pi}\int_{0}^{\pi} [f(x+0)+f(x-0)]\Psi_n(t)\,dt \qquad (47.13)$$

because of the evenness of $\Psi_n(t)$. From (47.12) and the evenness of $\Psi_n(t)$ we conclude that

$$f_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} [f(x+t)+f(x-t)]\Psi_n(t)\,dt. \qquad (47.14)$$

Therefore from (47.13) and (47.14)

$$f_n(x) - \frac{f(x+0)+f(x-0)}{2}$$
$$= \frac{1}{\pi}\int_{0}^{\pi} [f(x+t)+f(x-t)-f(x+0)-f(x-0)]\Psi_n(t)\,dt. \qquad (47.15)$$

We will show that the integral on the right-hand side of (47.15) tends to zero as $n \to \infty$ and moreover if $f(x)$ is continuous in (a, b), then it tends to zero uniformly in $[\alpha, \beta]$, where $a < \alpha < \beta < b$. For this purpose we will choose a number δ such that

$$\begin{matrix} |f(x+t)-f(x+0)| < \varepsilon, \\ |f(x-t)-f(x-0)| < \varepsilon \end{matrix} \quad \text{at} \quad 0 \leqslant x \leqslant \delta. \qquad (47.16)$$

This is possible for any fixed x; if $f(x)$ is continuous in (a, b) (in this case $f(x+0) = f(x-0) = f(x)$), then it is possible to choose δ so that it is independent of x, $\alpha \leqslant x \leqslant \beta$ and the inequalities (47.16) hold. Having chosen δ in this way, we divide the integral of formula (47.15) into two: integral I_1 in the interval $(0, \delta)$ and integral I_2 in the interval (δ, π). We have on the basis of (47.16)

$$|I_1| < 2\varepsilon \int_0^{\pi} |\Psi_n(t)|\,dt < 2\varepsilon C$$

from property (2) of the function $\Psi_n(t)$.

For I_2 we find

$$|I_2| \leqslant M_n(\delta) \int_{\delta}^{\pi} \{|f(x+t)| + |f(x+0)| + |f(x-t)| + |f(x-0)|\}dt. \qquad (47.17)$$

For constant x the integral in (47.17) is finite and the factor in front of it tends to zero because of property (3) of the function $\Psi_n(t)$, which means that $I_2 \to 0$. Moreover, if $x \in [\alpha, \beta] \subset (a, b)$, then the integral in (47.17) for any x does not exceed

$$\int_{-\pi}^{\pi} |f(t)|\, dt + 2\pi\, |f(x)|,$$

and since $f(x)$ is continuous in (a, b) and is therefore bounded in $[\alpha, \beta]$, then $I_2 \to 0$ uniformly. The lemma is completely proved.

In order to derive Fejér's theorem formulated above from the above lemma, it is sufficient to prove that the Fejér kernel satisfies the properties given in the lemma; then, supposing $f_n(x) = \sigma_n(x)$, we arrive at the required result.

But property (1) for a Fejér kernel has been fulfilled; (3) and (4) have already been proved by us and (2) follows from the fact that for a Fejér kernel

$$\int_{-\pi}^{\pi} |K_n(t)|\, dt = \int_{-\pi}^{\pi} K_n(t)\, dt = \pi$$

since $K_n(t) \geqslant 0$ and from property (3). Thus Fejér's theorem is completely proved.

§ 48. Corollaries of Fejér's theorem

From Fejér's theorem, it is possible to deduce a number of interesting corollaries. First, it gives a new proof of Weierstrass's classic theorem on the approximation of a continuous function by means of a trigonometric polynomial (see § 27).

Indeed, since we have proved that for a continuous $f(x)$ the function $\sigma_n(x)$ tends uniformly to $f(x)$, then having chosen n sufficiently large, it can be stated that

$$|f(x) - \sigma_n(x)| < \varepsilon, \quad -\infty < x < +\infty.$$

But $\sigma_n(x)$ is evidently a trigonometric polynomial and therefore the theorem is proved.

We will also note that method $(C, 1)$ is regular (see Introductory Material, § 6) i.e. the convergence of a series to a value S implies its summability by method $(C, 1)$ to the same value S. From this, it immediately follows that:

If $\sigma(f)$ converges at a point of continuity of the function $f(x)$, then it converges to $f(x)$; similarly, at a point of discontinuity of the first kind, if $\sigma(f)$ converges, then it certainly tends to $[f(x+0) + f(x-0)]/2$.

Finally, Fejér sums make it possible in some cases to pass judgment on the normal partial sums of the Fourier series. Thus, for example, it is possible to prove the theorem:

For the function $f(x)$ of bounded variation, the partial sums of the series $\sigma(f)$ are all bounded.

In order to prove this, we will note first that if

$$m \leqslant f(x) \leqslant M, \quad -\pi \leqslant x \leqslant \pi, \tag{48.1}$$

then for Fejér sums we also have

$$m \leqslant \sigma_n(x) \leqslant M, \quad -\pi \leqslant x \leqslant \pi. \tag{48.2}$$

Indeed, taking into account that the Fejér kernel is positive, we immediately derive from (47.4) and (48.1) that

$$m\frac{1}{\pi}\int_{-\pi}^{\pi} K_n(t)\,dt \leqslant \sigma_n(x) \leqslant M\frac{1}{\pi}\int_{-\pi}^{\pi} K_n(t)\,dt,$$

and then formula (47.10) immediately shows the truth of our statement.

Having noted this, we will now compare $\sigma_n(x)$ and $S_n(x)$. We have (see Introductory Material, § 6)

$$S_n(x) - \sigma_n(x) = \frac{1}{n+1}\sum_{k=1}^{n} k(a_k \cos kx + b_k \sin kx). \tag{48.3}$$

Hence it follows that

$$|S_n(x) - \sigma_n(x)| \leqslant \frac{1}{n+1}\sum_{k=1}^{n} k(|a_k| + |b_k|).$$

But if $f(x)$ is of bounded variation, then as we know (§ 22)

$$|a_k| \leqslant \frac{V}{K} \quad \text{and} \quad |b_k| \leqslant \frac{V}{K},$$

where V is the complete variation of $f(x)$; therefore

$$|S_n(x) - \sigma_n(x)| \leqslant 2V,$$

whence

$$2V - M \leqslant S_n(x) \leqslant 2V + M. \tag{48.4}$$

Formula (48.4) not only proves that the partial sums of the Fourier series for functions of bounded variation are all bounded, but it also gives the bounds within which they are contained in terms of the bounds of this function and its complete variation.

Note. We have seen (see (48.1) and (48.2)) that if $f(x)$ is contained between m and M in an interval of length 2π, then $\sigma_n(x)$ ($n = 1, 2, \ldots$) are also contained in this interval between m and M. Later we will find it useful to estimate $\sigma_n(x)$, knowing only the bounds of $f(x)$ in some interval $[a, b]$. We will prove that:

If

$$m \leqslant f(x) \leqslant M \quad \textit{in} \quad a \leqslant x \leqslant b,$$

then for any $\delta > 0$, N_0 *(dependent on* δ*) is found such that*

$$m - \delta \leqslant \sigma_n(x) \leqslant M + \delta \quad \text{at} \quad n \geqslant N_0(\delta), \quad a + \delta \leqslant x \leqslant b - \delta. \tag{48.5}$$

Indeed, from (47.4) we find that

$$\sigma_n(x) = \frac{1}{\pi}\int_{-\pi}^{-\delta} f(x+u)K_n(u)\,du + \frac{1}{\pi}\int_{-\delta}^{\delta} f(x+u)\,K_n(u)\,du$$
$$+ \frac{1}{\pi}\int_{\delta}^{\pi} f(x+u)K_n(u)\,du = I_n' + I_n'' + I_n'''. \tag{48.6}$$

From (47.7), it follows that

$$I_n' = O\left(\frac{1}{n}\right)\int_{-\pi}^{-\delta} |f(x+u)|\, du = O\left(\frac{1}{n}\right)\int_{-\pi}^{\pi} |f(u)|\, du = o(1) \tag{48.7}$$

and a similar result holds for I_n'''.

To estimate I_n'' we note that if $a + \delta \leqslant x \leqslant b - \delta$ and $|u| \leqslant \delta$, then $x + u \in [a, b]$ and then

$$m\frac{1}{\pi}\int_{-\delta}^{\delta} K_n(u)\, du \leqslant I_n'' \leqslant M\frac{1}{\pi}\int_{-\delta}^{\delta} K_n(u)\, du.$$

But we know from (47.11) that

$$\lim_{n\to\infty} \frac{1}{\pi}\int_{-\delta}^{\delta} K_n(u)\, du = 1.$$

This means that it is possible to choose N_0 (dependent on δ) so large that, for example, we have

$$m - \frac{\delta}{3} \leqslant I_n'' \leqslant M + \frac{\delta}{3}, \quad n \geqslant N_0,$$

and moreover (see (48.7))

$$|I_n'| \leqslant \frac{\delta}{3} \quad \text{and} \quad |I_n'''| < \frac{\delta}{3},$$

whence we see from (48.6) that (48.5) is proved.

§ 49. Fejér–Lebesgue theorem

Fejér's theorem, proved in § 47, makes it possible to judge the summability of the series $\sigma(f)$ only at those points where $f(x)$ is either continuous or possesses a discontinuity of the first kind. However, an arbitrary summable function cannot possess a point of the given type. Lebesgue generalized Fejér's result and proved the following theorem.

FEJÉR–LEBESGUE THEOREM. *For any summable function $f(x)$, the series $\sigma(f)$ is summable almost everywhere by Fejér's method to $f(x)$.*

To prove this theorem, let us assume

$$\varphi_x(t) = f(x+t) + f(x-t) - 2f(x) \tag{49.1}$$

and

$$\Phi_x(t) = \int_0^t |\varphi_x(u)|\, du. \tag{49.2}$$

We will prove that the series $\sigma(f)$ is summable by Fejér's method to $f(x)$ at any point x where

$$\Phi_x(t) = o(t). \tag{49.3}$$

For this we note (see § 47) that

$$\sigma_n(x) - f(x) = \frac{1}{\pi}\int_0^{\pi} [f(x+t) + f(x-t) - 2f(x)]K_n(t)\,dt$$

$$= \frac{1}{\pi}\int_0^{\pi} \varphi_x(t)K_n(t)\,dt \tag{49.4}$$

and we will prove that when (49.3) is fulfilled the integral on the right-hand side of (49.4) tends to zero. For this purpose we note that

$$|K_n(t)| \leqslant 2n \quad (n \geqslant 1), \tag{49.5}$$

since

$$|D_k(t)| \leqslant k + \frac{1}{2} < 2n \quad \text{for any} \quad k \leqslant n,$$

and

$$K_n(t) = \frac{1}{n+1}\sum_{k=0}^{n} D_k(t).$$

Therefore

$$\left|\frac{1}{\pi}\int_0^{1/n} \varphi_x(t)K_n(t)\,dt\right| \leqslant \frac{2n}{\pi}\int_0^{1/n} |\varphi_x(t)|\,dt = \frac{2n}{\pi}\Phi_x\left(\frac{1}{n}\right) = o(1) \tag{49.6}$$

due to (49.3).

Also, because of (47.6)

$$\left|\frac{1}{\pi}\int_{1/n}^{\pi} \varphi_x(t)K_n(t)\,dt\right| \leqslant \frac{\pi}{2n}\int_{1/n}^{\pi} |\varphi_x(t)|\,\frac{dt}{t^2}. \tag{49.7}$$

For the integral on the right-hand side of (49.7) we will carry out integration by parts; we obtain, again operating on (49.3) (see Introductory Material, § 11)

$$\frac{\pi}{2n}\int_{1/n}^{\pi} |\varphi_x(t)|\,\frac{dt}{t^2} = \frac{\pi}{2n}\left[\frac{\Phi_x(\pi)}{\pi^2} - \frac{\Phi_x\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)^2}\right] + \frac{2\pi}{2n}\int_{1/n}^{\pi} \Phi_x(t)\,\frac{dt}{t^3}$$

$$= o(1) + \frac{1}{n}\,o\left(\int_{1/n}^{\pi} \frac{dt}{t^2}\right) = o(1). \tag{49.8}$$

From (49.6) and (49.8), it follows because of (49.4) that

$$\sigma_n(x) - f(x) = o(1)$$

at every point, where (49.3) is fulfilled.

It remains for us to prove that the condition (49.3) holds almost everywhere. But in § 15 of the Introductory Material it was remarked that this relationship is fulfilled at any Lebesgue point and consequently almost everywhere.

The Fejér–Lebesgue theorem is proved.

As a corollary, we obtain the following important theorem.

If $\sigma(f)$ converges in some set E, $mE > 0$, then its sum equals $f(x)$ almost everywhere in E.

Indeed, we know that method $(C, 1)$ is regular. Therefore at the point where $\sigma(f)$ possesses a certain sum S, it should be summable to this value S by Fejér's method. But since by Fejér's method it is summable to $f(x)$ almost everywhere, then the set of points of E, where the sum of the series $\sigma(f)$ differs from $f(x)$, is of measure zero.

Note. We have seen that the series $\sigma(f)$ is summable by Fejér's method at any Lebesgue point. It is known that a these points $f(x)$ is the derivative of its indefinite integral. The question can be raised whether the series $\sigma(f)$ is summable by Fejér's method at a point, where the latter condition is fulfilled. Lebesgue[1] proved that this should not, however, occur, though here we have summability $(C, 2)$.

§ 50. Estimate of the partial sums of a Fourier series

In § 49 we have proved that at the points where the following condition is fulfilled,

$$\Phi_x(h) = \int_0^h |f(x+u) + f(x-u) - 2f(x)|\, du = o(h), \tag{50.1}$$

the series $\sigma(f)$ is summable by Fejér's method. It was also remarked that condition (50.1) is fulfilled almost everywhere. Now we want to estimate the increase of the partial sums $S_n(x)$ at these points.

We will prove that at any point x, where (50.1) is fulfilled, we have

$$S_n(x) = o(\ln n). \tag{50.2}$$

Consequently, estimate (50.2) also holds almost everywhere.

We have seen (see (37.9)) that

$$S_n(x) - f(x) = \frac{1}{\pi}\int_0^\delta [f(x+u) + f(x-u) - 2f(x)]\frac{\sin nu}{u}\, du + o(1), \tag{50.3}$$

where $o(1)$ tends to zero and δ is any positive given number. Supposing $\varphi(u) = f(x+u) + f(x-u) - 2f(x)$, we have

$$\int_0^\delta |\varphi(x)| \left|\frac{\sin nu}{u}\right| du = \int_0^{1/n} |\varphi(u)| \frac{\sin nu}{u}\, du + \int_{1/n}^\delta |\varphi(u)| \left|\frac{\sin nu}{u}\right| du$$

$$\leqslant n\int_0^{1/n} |\varphi(u)|\, du + \int_{1/n}^\delta |\varphi(u)| \frac{1}{u}\, du. \tag{50.4}$$

Then from (50.1)

$$\Phi_x(h) = \int_0^h |\varphi(u)|\, du, \tag{50.5}$$

therefore (for brevity's sake, we will dispense with the index x)

$$\int_0^{1/n} |\varphi(u)|\, du = \Phi\left(\frac{1}{n}\right) = o\left(\frac{1}{n}\right) \tag{50.6}$$

and

$$\int_{1/n}^{\delta} |\varphi(u)| \frac{du}{u} = \frac{\Phi(t)}{t}\Bigg|_{1/n}^{\delta} + \int_{1/n}^{\delta} \frac{\Phi(t)}{t^2}\, dt. \tag{50.7}$$

From (50.4), (50.6) and (50.7) it follows that

$$\int_0^{\delta} |\varphi(u)| \left|\frac{\sin nu}{u}\right| du \leqslant n\Phi\left(\frac{1}{n}\right) + \frac{\Phi(\delta)}{\delta} + n\Phi\left(\frac{1}{n}\right) + \int_{1/n}^{\delta} \frac{\Phi(t)}{t^2}\, dt$$

$$= O(1) + \int_{1/n}^{\delta} \frac{\Phi(t)}{t^2}\, dt. \tag{50.8}$$

If for a given $\varepsilon > 0$ we choose δ so that $\Phi(t) < \varepsilon t$ for $0 \leqslant t \leqslant \delta$, which is possible from (50.1), then

$$\int_{1/n}^{\delta} \frac{\Phi(t)}{t^2}\, dt < \varepsilon \int_{1/n}^{\delta} \frac{dt}{t} = \varepsilon \ln n\delta = o(\ln n), \tag{50.9}$$

because ε is as small as desired. But $o(1)$ is also $o(\ln n)$, therefore from (50.3), (50.8) and (50.9) we find that

$$|S_n(x) - f(x)| = o(\ln n).$$

But since x is fixed, then $f(x)$ is constant, i.e. $|f(x)| = o(\ln n)$ and finally

$$S_n(x) = o(\ln n),$$

which is what was required to be proved.

Note. In § 36 it was proved that for a bounded function, which also means a continuous function, we have for all x and $n > 1$

$$|S_n(x)| \leqslant CM \ln n \quad (n = 2, 3, \ldots),$$

if $|f(x)| \leqslant M$ (C is an absolute constant). If $f(x)$ is continuous, then condition (50.1) is fulfilled and even uniformly; therefore, for continuous functions the estimate made earlier is replaced by a stronger one: $O(\ln n)$ is replaced by $o(\ln n)$.

§ 51. Convergence factors

It is usually said that the numbers $\{\mu_n\}$ are *convergence factors* for some series

$$u_0(x) + u_1(x) + \cdots + u_n(x) + \cdots$$

in the interval $[a, b]$, if the series

$$\sum u_n(x)\,\mu_n$$

converges almost everywhere in $[a, b]$.

The results of §§ 49 and 50 allow us to prove that it is possible to choose as convergence factors for a Fourier series in $[-\pi, \pi]$ the numbers

$$\mu_n = \frac{1}{\ln n}, \quad n = 2, 3, \ldots$$

(μ_0 and μ_1 can be chosen as desired); i.e. we have

THEOREM. *If a_k and b_k are Fourier coefficients ($k = 1, 2, \ldots$) then the series*

$$\sum_{k=2}^{\infty} \frac{a_k \cos kx + b_k \sin kx}{\ln k}$$

converges almost everywhere in $[-\pi, \pi]$.

To prove this we note that in § 50 it was proved that $S_n(x) = o(\ln n)$ almost everywhere. Therefore because the sequence μ_n is convex (the definition and properties of convex sequences are given in § 3 of Introductory Material), then it remains to apply Theorem 6 (see Appendix, § 12), assuming that $u_n(x) = a_n \cos nx + b_n \sin nx$.

§ 52. Comparison of Dirichlet and Fejér kernels

We know (see §§ 45 and 46) that continuous functions exist in which the Fourier series diverges at some point. On the other hand, for any continuous function $f(x)$, the series $\sigma(f)$ is summable to $f(x)$ at any point (see § 47).

We want to explain why such a phenomenon occurs and for this purpose we will compare the Dirichlet and Fejér kernels. As is known

$$S_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, D_n(t - x)\, dt \tag{52.1}$$

and

$$\sigma_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, K_n(t - x)\, dt, \tag{52.2}$$

where $D_n(u)$ is a Dirichlet kernel and $K_n(u)$ is a Fejér kernel.

If at the point x_0, the series $\sigma(f)$ converges to $f(x)$, then this means that $S_n(x_0) \to f(x_0)$; if it is summable by Fejér's method to $f(x)$, then $\sigma_n(x_0) \to f(x_0)$.

It is natural, therefore, to pose the question thus: let $f(x)$ be continuous and

$$f_n(x) = \int_{-\pi}^{\pi} f(t)\,\Phi_n(t - x)\,dt, \tag{52.3}$$

where $\Phi_n(u)$ is some function which we will also refer to as a *kernel;* we ask ourselves—what properties of this kernel influence the equality

$$\lim_{n\to\infty} f_n(x) = f(x)$$

or the existence of points x_0 where $f_n(x_0)$ does not tend to $f(x_0)$ or in general does not tend to any limit?

Before answering this question we will show that the problem of convergence of a Fourier series with respect to an arbitrary orthogonal system leads to another question of the same type and we will therefore solve both problems together.

Let $\{\varphi_n(x)\}$ be some orthonomal system in (a, b). In order to study the convergence of a Fourier series for some function $f(x)$ with respect to this system, we will consider the partial sum of this series, i.e.

$$S_n(x) = \sum_{k=0}^{n} a_k \varphi_k(x),$$

in other words,

$$S_n(x) = \sum_{k=0}^{n} \varphi_k(x) \int_a^b f(t)\varphi_k(t)\,dt = \int_a^b f(t)\left[\sum_{k=0}^{n} \varphi_k(t)\varphi_k(x)\right] dt.$$

Supposing

$$\Phi_n(t, x) = \sum_{k=0}^{n} \varphi_k(t)\varphi_k(x),$$

we name the function $\Phi_n(t, x)$ the kernel of the system $\{\varphi_n(x)\}$. We have

$$S_n(x) = \int_a^b f(t)\Phi_n(t, x)\,dt. \tag{52.4}$$

Lebesgue was the first to pay attention to the importance of investigating the behaviour of functions of the type

$$\varrho_n(x) = \int_a^b |\Phi_n(t, x)|\,dt. \tag{52.5}$$

which are now usually called "*Lebesgue functions*" for the given system. The role of these functions in the problem of the convergence of a Fourier series becomes extremely clear, when the theorem is proved (see Lebesgue[21]).

THEOREM. *If for some point x_0 the sequence $\varrho_n(x_0)$ $(n = 1, 2, \ldots)$ is unbounded, then there exists a continuous function $f(x)$ for which the Fourier series with respect to the system $\{\varphi_n(x)\}$ diverges without bound at the point x_0.*

This theorem can be proved immediately, if we first establish the validity of the following more general assertion:

LEMMA. *Let*

$$f_n(x, f) = \int_a^b f(t)\Phi_n(t, x)\,dt, \tag{52.6}$$

where $\Phi_n(t, x)$ for every fixed x is summable with respect to the variable t and $f(t)$ is bounded. Then, if

$$\overline{\lim_{n\to\infty}}\, \varrho_n(x_0) = +\infty, \tag{52.7}$$

a continuous function $f(x)$ is found for which

$$\overline{\lim_{n\to\infty}}\, |f_n(x_0, f)| = +\infty. \tag{52.8}$$

Indeed, first, supposing that for a given n

$$g(t) = \operatorname{sign}\Phi_n(t, x_0),$$

we have

$$f_n(x_0, g) = \int_a^b g(t)\Phi_n(t, x_0)\, dt = \int_a^b |\Phi_n(t, x_0)|\, dt = \varrho_n(x_0). \tag{52.9}$$

This means that for any n there exists a function $g(t)$ such that $|g(t)| \leqslant 1$ and for it

$$f_n(x_0, g) = \varrho_n(x_0). \tag{52.10}$$

If this were the same function $g(t)$ for all n and if it were continuous, the theorem would be proved, because from (52.7) we would have

$$\overline{\lim_{n\to\infty}}\, |f_n(x_0, g)| = +\infty.$$

Therefore, we will first replace $g(t)$ by a continuous function $g^*(t)$, for which $f_n(x_0, g^*)$ is "large" and then we will transfer various functions for various n to a single function.

We will first choose for a given n a continuous $g^*(t)$ such that for $f_n(x, g^*)$ we have

$$f_n(x_0, g^*) \geqslant \tfrac{1}{2}\varrho_n(x_0). \tag{52.11}$$

For this it is sufficient to take ε such that

$$\int_E |\Phi_n(t, x_0)|\, dt \leqslant \frac{1}{4}\varrho_n(x_0), \tag{52.12}$$

if $mE < \varepsilon$, which is possible, since for fixed n and x_0 the function under the integral sign is a summable function of t, and therefore its integral is as small as desired, if the set over which the integration occurs is of sufficiently small measure. Because of the C-property we can find a continuous function $g^*(t)$ coinciding with $g(t)$ in the perfect set P, $mP > (b - a) - \varepsilon$, such that $|g^*(t)| \leqslant 1$. Then for this function from (52.6)

$$f_n(x_0, g^*) = \int_a^b g^*(t)\Phi_n(t, x_0)\, dt.$$

From (52.12) it follows that

$$\begin{aligned} |f_n(x_0, g^*) - f_n(x_0, g)| &= \Big|\int_{CP} [g^*(t) - g(t)]\Phi_n(t, x_0)\, dt\Big| \\ &\leqslant 2\int_{CP} |\Phi_n(t, x_0)|\, dt \leqslant \frac{1}{2}\varrho_n(x_0) \end{aligned} \tag{52.13}$$

which means that (52.11) follows from (52.9) and (52.13).

For all n we will denote by $g_n(t)$ the function which possesses the properties:

(a) $g_n(t)$ is continuous
(b) $|g_n(t)| \leqslant 1$
(c) $f_n(x_0, g_n) > \frac{1}{2}\varrho_n(x_0)$.

We have already seen that it is possible to establish such a function for all n. Now let ε_n be a sequence of numbers such that

$$\varepsilon_n > 0, \sum_{n=1}^{\infty} \varepsilon_n < +\infty, \sum_{k=n+1}^{\infty} \varepsilon_k \leqslant \frac{1}{6}\varepsilon_n \tag{52.14}$$

(for example, it is possible to take $\varepsilon_n = 1/7^n$), let n_k be an increasing sequence of integers which we will select later. Then, supposing

$$f(x) = \sum_{k=1}^{\infty} \varepsilon_k g_{n_k}(x), \tag{52.15}$$

we see that $f(x)$ is continuous, since $g_n(x)$ are continuous and all $|g_n(x)| \leqslant 1$ and $\sum \varepsilon_k < +\infty$, which means that the series (52.15) converges uniformly. It is clear that

$$f_n(x_0, f) = \int_a^b \sum_{k=1}^{\infty} \varepsilon_k g_{n_k}(t)\Phi_n(t, x_0)\, dt = \sum_{k=1}^{\infty} \varepsilon_k \int_a^b g_{n_k}(t)\Phi_n(t, x_0)\, dt$$

$$= \sum_{k=1}^{\infty} \varepsilon_k f_n(x_0, g_{n_k}).$$

Here the term-by-term integration is valid because of the uniform convergence of the series (52.15).

We will now show that for a suitable choice of the numbers n_k we will have

$$\overline{\lim_{n\to\infty}} |f_n(x_0, f)| = +\infty. \tag{52.8}$$

If for even one of the functions $g_m(x)$ we had

$$\overline{\lim_{n\to\infty}} |f_n(x_0, g_m)| = +\infty,$$

then the theorem would be proved. We will assume that this is not the case. Let us define

$$\overline{\lim_{n\to\infty}} |f_n(x_0, g_m)| = \gamma_m. \tag{52.16}$$

We will choose by induction the numbers n_k such that

$$\varepsilon_k \varrho_{n_k}(x_0) \to \infty \tag{52.17}$$

and

$$\sum_{p=1}^{k-1} \varepsilon_p \gamma_{n_p} \leqslant \frac{1}{12} \varepsilon_k \varrho_{n_k}(x_0). \tag{52.18}$$

This is possible, since $\{\varrho_n(x_0)\}$ is unbounded due to (52.7), which means that the numbers n_k can be chosen such that $\varrho_{n_k}(x_0) \to \infty$ sufficiently quickly for conditions (52.17) and (52.18) to be fulfilled. Since

$$\left| f_n \left(x_0, \sum_{p=1}^{k-1} \varepsilon_p g_{n_p} \right) \right| = \left| \sum_{p=1}^{k-1} \varepsilon_p f_n(x_0, g_{n_p}) \right| < 2 \sum_{p=1}^{k-1} \varepsilon_p \gamma_{n_p}$$

as $n \to \infty$ due to (52.16), it is possible to choose n_k so large that

$$\left| f_{n_k} \left(x_0, \sum_{p=1}^{k-1} \varepsilon_p g_{n_p} \right) \right| < 2 \sum_{p=1}^{k-1} \varepsilon_p \gamma_{n_p} \leqslant \frac{1}{6} \varepsilon_k \varrho_{n_k}(x_0) \tag{52.19}$$

because of (52.18).

On the other hand

$$\left| f_{n_k} \left(x_0, \sum_{p=k+1}^{\infty} \varepsilon_p g_{n_p} \right) \right| < \sum_{p=k+1}^{\infty} \varepsilon_p \varrho_{n_k}(x_0) \leqslant \frac{1}{6} \varepsilon_k \varrho_{n_k}(x_0), \tag{52.20}$$

because $|g_{np}(x)| \leqslant 1$ which means that

$$\left| \int_a^b g_{n_p}(t) \Phi_{n_k}(t, x_0)\, dt \right| \leqslant \int_a^b |\Phi_{n_k}(t, x_0)|\, dt = \varrho_{n_k}(x_0)$$

and moreover, we have (52.14).

Hence because of property (c) of the function $g_n(x)$, (52.19) and (52.20)

$$|f_{n_k}(x_0, f)| \geqslant f_{n_k}(x_0, \varepsilon_k g_{n_k}) - \left| f_{n_k} \left(x_0, \sum_{p=1}^{k-1} \varepsilon_p g_{n_p} \right) \right| - \left| f_{n_k} \left(x_0, \sum_{k+1}^{\infty} \varepsilon_p g_{n_k} \right) \right|$$

$$\geqslant \frac{1}{2} \varepsilon_k \varrho_{n_k}(x_0) - \frac{1}{6} \varepsilon_k \varrho_{n_k}(x_0) - \frac{1}{6} \varepsilon_k \varrho_{n_k}(x_0) \geqslant \frac{1}{6} \varepsilon_k \varrho_{n_k}(x_0),$$

and this tends to $+\infty$ as $k \to \infty$ due to (52.17). This means that (52.8) is valid and the theorem is proved.†

The validity of Lebesgue's theorem formulated above follows quickly from this. Indeed, if in the proved lemma the role played by $\Phi_n(t, x)$ is the kernel of the given orthogonal system, then $f_n(t, f)$ is converted into the partial sum of the Fourier series of $f(x)$ with respect to this system (due to (52.4) and (52.3)) and therefore, if at some point (52.7) is fulfilled, then a continuous $f(x)$ is found with a Fourier series which diverges at this point. Thus, Lebesgue's theorem is proved.

Let us now consider specially the case of a trigonometric system. If it is normalized, i.e. if the following system is taken

$$\frac{1}{\sqrt{2\pi}}, \quad \frac{\cos x}{\sqrt{\pi}}, \quad \frac{\sin x}{\sqrt{\pi}}, \ldots, \frac{\cos nx}{\sqrt{\pi}}, \quad \frac{\sin nx}{\sqrt{\pi}}, \ldots,$$

---

† Since on multiplying $f(t)$ by some constant $f_n(x, f)$ is multiplied by the same constant by virtue of (52.6), it is always possible to find $f(x)$ to satisfy the conditions of the lemma and such that $|f(t)| \leqslant 1$. This note is not necessary for Lebesgue's theorem but will be useful later in Chapter IV.

then the role of its kernel is played by the function

$$\Phi_n(t, x) = \frac{1}{2\pi} + \sum_{k=1}^{n} \frac{\cos kx \cos kt + \sin kx \sin kt}{\pi}$$
$$= \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{n} \cos k(t - x) = \frac{1}{\pi} D_n(t - x),$$

and therefore the Lebesgue functions (see (52.5)) have the form

$$\varrho_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} |D_n(t - x)|\, dt.$$

But because of the periodicity of $D_n(u)$ we have

$$\varrho_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} |D_n(t)|\, dt,$$

i.e. the Lebesgue functions do not depend on x and are converted into the Lebesgue constants L_n considered earlier (see § 35). But we know that $\lim\limits_{n\to\infty} L_n = +\infty$ (because $L_n \approx (4/\pi^2)\ln n$) and therefore we now see that the existence of continuous functions with Fourier series, divergent at some point, is explained by the fact that the Lebesgue constants increase without bound with increase in n. We also note that since

$$\varrho_n(x) = L_n$$

for any x, then it is possible for any point x to find a continuous $f(x)$ with a Fourier series divergent at this point.

Now we will return to the question of the summability of a Fourier series by Fejér's method. Comparing formulae (52.5) and (52.2), we see that if for

$$\varrho_n(t) = \frac{1}{\pi} \int_{-\pi}^{\pi} |K_n(t - x)|\, dt$$

(52.7) were fulfilled for even one value of x_0, then it would be possible to find a continuous $f(x)$ for which $\sigma_n(x,f)$ would not tend to any finite limit as $n \to \infty$, i.e. $\sigma(f)$ would be unsummable by Fejér's method at this point. But due to $K_n(u)$ being periodic and positive, we have

$$\varrho_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} K_n(t)\, dt,$$

and then due to property (3) of Fejér kernels (see § 47)

$$\varrho_n(x) \equiv 1$$

for all n and x. Thus, for a Fejér kernel the fulfilment of (52.7) at no point whatever is impossible.

In § 2 of Chapter VII we will see why Fejér's method is applicable almost everywhere (Fejér–Lebesgue theorem, § 49) whilst everywhere divergent Fourier series exist (Chapter V, § 20) – this is also the result of the different behaviour of Fejér and Dirichlet kernels.

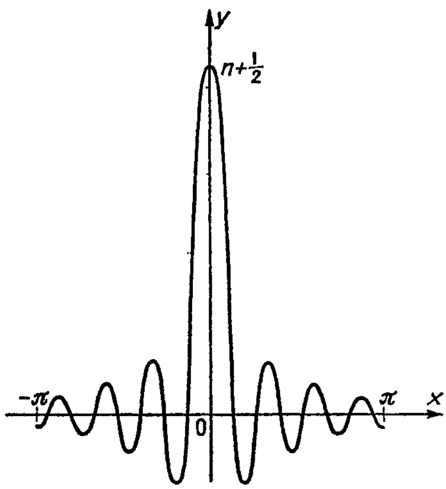

FIG. 9

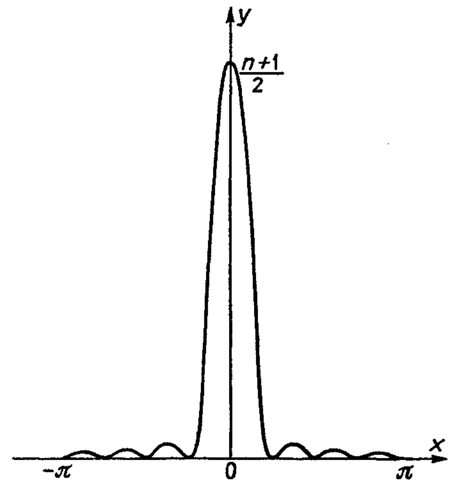

FIG. 10

To conclude this section, we think it appropriate to represent Dirichlet and Fejér kernels geometrically (see Figs. 9 and 10).

§ 53. Summation of Fourier series by the Abel–Poisson method

We will refer here to yet another classic and very important method of summation of Fourier series. For this we recall (see Introductory Material, § 7) that the series $\sum_{n=0}^{\infty} u_n(x)$ is said to be summable by Abel's method at a point x_0 to the value S, if for any r, $0 \leqslant r < 1$, the series $\sum_{n=0}^{\infty} u_n(x_0)r^n$ converges and supposing

$$S(x_0, r) = \sum_{n=0}^{\infty} u_n(x_0)r^n,$$

we have

$$\lim_{r\to 1} S(x_0, r) = S.$$

Poisson applied this method of summation to Fourier series, therefore the given method when it is applied to trigonometric series is usually referred to as *Poisson's method or the Abel–Poisson method.*

Since we know (see Introductory Material, § 7) that Abel's method is stronger than the method $(C, 1)$, then the following theorem immediately results from Fejér's theorem and the Fejér–Lebesgue theorem (see §§ 47 and 49):

THEOREM. *For any summable $f(x)$ the series $\sigma(f)$ is summable almost everywhere by the Abel–Poisson method to this function $f(x)$; it is summable to $\frac{1}{2}[f(x+0)+f(x-0)]$ at any point of discontinuity of the first kind and to $f(x)$ at any point of continuity.*

It can be seen that apart from these theorems little more need be said concerning the summation of Fourier series by Poisson's method; however, we will see in § 55 and § 56 that it is possible to obtain very much deeper results. We will first derive some auxiliary formulae which will be necessary for us there.

For any trigonometric series

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{53.1}$$

"Poisson sums" are the names given to the functions

$$f(r, x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) r^n, \tag{53.2}$$

when the series on the right-hand side of (53.2) converges. In the case when the series (53.1) is a Fourier series for some function $f(x)$, these functions can be expressed in terms of $f(x)$ in the integral form, in the same way as was done for the partial sums and Fejér sums of a Fourier series. We will find this in the next section. Also in § 57 we will use the results obtained to solve an important problem, called Dirichlet's problem.

§ 54. Poisson kernel and Poisson integral

We will first find a suitable expression for $f(r, x)$ if (53.1) is $\sigma(f)$. We have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos nt\, dt, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin nt\, dt,$$

and therefore

$$f(r, x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, dt + \frac{1}{\pi}\sum_{n=1}^{\infty} r^n \int_{-\pi}^{\pi} f(t) \cos n(t-x)\, dt.$$

But since $0 \leqslant r < 1$, then the series $\sum_{n=1}^{\infty} r^n \cos n(t-x)$ for a given r converges uniformly with respect to t and therefore according to Lebesgue's theorem (Introductory Material, § 14) it is possible to integrate it term-by-term even after multiplying by $f(x)$; therefore

$$f(r, x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2}+\sum_{n=1}^{\infty} r^n \cos n(t-x)\right] dt. \tag{54.1}$$

Let us now find a simpler expression for the series given in the square brackets in (54.1). Let

$$P(r, \alpha) = \frac{1}{2}+\sum_{n=1}^{\infty} r^n \cos n\alpha.$$

We consider the auxiliary series

$$\frac{1}{2}+\sum_{n=1}^{\infty} z^n$$

and suppose that $z = r(\cos\alpha + i\sin\alpha)$. If $|z| = r < 1$, then this series converges and

$$\frac{1}{2}+\sum_{n=1}^{\infty} z^n = \frac{1}{2}+\frac{z}{1-z} = \frac{1+z}{2(1-z)} = \frac{1-r^2+2ir\sin\alpha}{2[1-2r\cos\alpha+r^2]}.$$

But, on the other hand

$$\frac{1}{2}+\sum_{n=1}^{\infty} z^n = \frac{1}{2}\sum_{n=1}^{\infty} r^n(\cos n\alpha + i\sin n\alpha).$$

Therefore, separating the real and purely imaginary parts, we find

$$\frac{1}{2}+\sum_{n=1}^{\infty} r^n\cos n\alpha = \frac{1-r^2}{2[1-2r\cos\alpha+r^2]}$$

and

$$\sum_{n=1}^{\infty} r^n\sin n\alpha = \frac{r\sin\alpha}{1-2r\cos\alpha+r^2}.$$

Thus we have established that

$$P(r,\alpha) = \frac{1-r^2}{2[1-2r\cos\alpha+r^2]}. \tag{54.2}$$

This expression is known as a *Poisson kernel* and the expression

$$Q(r,\alpha) = \frac{r\sin\alpha}{1-2r\cos\alpha+r^2} \tag{54.3}$$

as *the kernel conjugate to it.*

Later, the fact that the Poisson kernel at $0 \leqslant r < 1$ is a positive value (as is also the Fejér kernel) will be very important. In fact, since

$$1-r^2>0 \quad\text{and}\quad 1-2r\cos\alpha+r^2 = (1-r)^2+4r\sin^2\frac{\alpha}{2}>0,$$

then $P(r,\alpha) > 0$ at $0 \leqslant r < 1$.

Let us return to formula (54.1). We have

$$f(r,x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,P(r,t-x)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\frac{1-r^2}{1-2r\cos(t-x)+r^2}\,dt. \tag{54.4}$$

The integral on the right-hand side of (54.4) is known as a *Poisson integral.*

It is very important to understand the meaning of a Poisson kernel geometrically (see Fig. 11). For this purpose we will take a plane circle with centre at the origin and unit radius; if a radius is drawn through the point M with polar co-ordinates (r,ω) and the perpendicular is drawn to it, then denoting by Q one of its points of intersection with the circumference, we find

$$\overline{MQ}^2 = 1-r^2.$$

If P is a point with polar co-ordinates $(1, t)$, then

$$\overline{MP}^2 = 1 - 2r\cos(\omega - t) + r^2,$$

and therefore

$$\frac{1 - r^2}{1 - 2r\cos(\omega - t) + r^2} = \left(\frac{\overline{MQ}}{\overline{MP}}\right)^2.$$

Thus, we again see that the Poisson kernel is a positive magnitude, and the Poisson integral can be written in the form

$$f(r, \omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\left(\frac{\overline{MQ}}{\overline{MP}}\right)^2 dt.$$

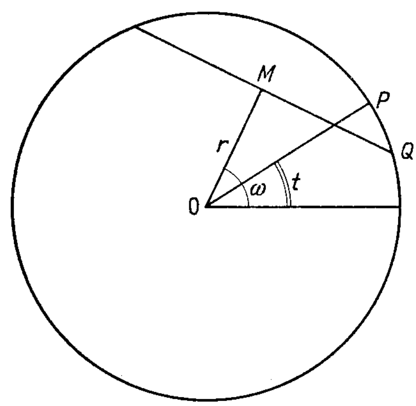

FIG. 11

Theorem § 53 could be expressed thus: if the point $M(r, \omega)$ tends to the point $P(1, \omega)$, i.e. to the point on the circumference lying on the same radius, then for almost all values of ω we have

$$f(r, \omega) \to f(\omega) \quad \text{as} \quad r \to 1$$

and this is true, in particular, for all those ω where $f(\omega)$ is continuous. But we want to prove that a considerably more general statement holds. We will now turn to this.

§ 55. Behaviour of the Poisson integral at points of continuity of a function

Let us prove the following theorem due to Fatou[1].

THEOREM. *If $f(\omega)$ is continuous at some point $P(1, \omega_0)$, then for the Poisson integral*

$$f(r, \omega) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\, P(r, \omega - t)\, dt \tag{55.1}$$

we have

$$f(r, \omega) \to f(\omega_0)$$

no matter how $M(r, \omega)$ tends to $P(1, \omega_0)$, provided it remains inside the circle of unit radius.

First we will note the following properties of a Poisson kernel:

(a) $P(r, t) \geqslant 0$ at any t and $0 \leqslant r < 1$.

(b) We have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} P(r, t)\, dt = 1.$$

Indeed from (54.4) supposing $f(t) \equiv 1$, we find that

$$1 = \frac{1}{\pi}\int_{-\pi}^{\pi} P(r, t - \omega)\, dt = \frac{1}{\pi}\int_{-\pi}^{\pi} P(r, t)\, dt. \tag{55.2}$$

(c) If $|t| \geqslant \delta$, then we have

$$m(r, \delta) = \max_{\delta \leqslant |t| \leqslant \pi} P(r, t) \to 0 \quad \text{as} \quad r \to 1. \tag{55.3}$$

Indeed

$$1 - 2r\cos t + r^2 \geqslant 1 - 2r\cos\delta + r^2 \quad \text{for} \quad \delta \leqslant |t| \leqslant \pi,$$

and therefore

$$0 \leqslant P(r, t) \leqslant \frac{1 - r^2}{2(1 - 2r\cos\delta + r^2)},$$

which also proves our statement.

From this and from (b) it immediately follows for any $\delta > 0$ that:

(d)

$$\lim_{r \to 1} \frac{2}{\pi}\int_{0}^{\delta} P(r, t)\, dt = 1. \tag{55.4}$$

Indeed, due to the evenness of $P(r, t)$ we have from (b)

$$1 = \frac{2}{\pi}\int_{0}^{\pi} P(r, t)\, dt = \frac{2}{\pi}\int_{0}^{\delta} P(r, t)\, dt + \frac{2}{\pi}\int_{\delta}^{\pi} P(r, t)\, dt,$$

and the latter integral does not exceed $(2/\pi)\, m(r, \delta)$.

Now in order to prove the theorem we note first that, multiplying (55.2) by $f(\omega_0)$, we have

$$f(\omega_0) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\omega_0)\, P(r, t - \omega)\, dt.$$

Subtracting this equation from (55.1) we find

$$f(r,\omega) - f(\omega_0) = \frac{1}{\pi}\int_{-\pi}^{\pi} [f(t) - f(\omega_0)]\, P(r, t-\omega)\, dt. \tag{55.5}$$

Let $\varepsilon > 0$ be given. We choose δ so that

$$|f(t) - f(\omega_0)| < \varepsilon \quad \text{for} \quad |t - \omega_0| < \delta, \tag{55.6}$$

and divide the integral (55.5) into three: for the range $\omega_0 - \delta < t < \omega_0 + \delta$ and for the ranges $(-\pi < t < \omega_0 - \delta)$ and $(\omega_0 + \delta < t < \pi)$. Due to the Poisson kernel being positive, and from (55.6) and (55.2) we have

$$\left| \frac{1}{\pi}\int_{\omega_0-\delta}^{\omega_0+\delta} [f(t) - f(\omega_0)]\, P(r, t-\omega)\, dt \right| < \frac{\varepsilon}{\pi}\int_{\omega_0-\delta}^{\omega_0+\delta} P(r, t-\omega)\, dt$$

$$< \frac{\varepsilon}{\pi}\int_{-\pi}^{\pi} P(r, t-\omega)\, dt = \varepsilon.$$

As regards the integrals in the remaining intervals, in them $|t - \omega| \geqslant \delta$ and therefore due to (55.3) it is possible to obtain

$$P(r, t-\omega) < \varepsilon,$$

provided r is taken sufficiently close to 1. Then the modulus of each of these integrals does not exceed

$$\frac{\varepsilon}{\pi}\int_{-\pi}^{\pi} [|f(t)| + |f(\omega_0)|]\, dt,$$

i.e. it can be made as small as desired.

The theorem is proved.

§ 56. Behaviour of a Poisson integral in the general case

We proved in § 55 that if $f(\omega)$ is continuous at $\omega = \omega_0$, then the Poisson integral tends to $f(\omega_0)$ independently of the path by which $M(r, \omega)$ tends to the point $P(1, \omega_0)$ (provided it remains inside the circle of unit radius).

In the case when $f(\omega)$ is not continuous at $\omega = \omega_0$, matters become more complicated. However, here it is possible to obtain good results only if M tends towards P not by any path but by non-tangential paths to the circle. This means that we permit the point M to move towards P provided it remains the whole time within some angle φ of magnitude $2\varphi < \pi$ with the bisector coinciding with OP (see Fig. 12).

Before studying the behaviour of the Poisson integral in the general case, we will prove a theorem by Fatou, concerning the behaviour of the partial derivative of $f(r, \omega)$ with respect to ω.

THEOREM 1. *If $f(\omega)$ possesses a finite derivative at the point $P(1, \omega_0)$, then*

$$\frac{\partial f(r, \omega)}{\partial \omega} \to f'(\omega_0),$$

if the point $M(r, \omega) \to P(1, \omega_0)$ by any non-tangential path.

In order to prove this, we will first prove a lemma.

LEMMA 1. *Let $f(u)$ have a bounded derivative $f'(\omega)$ in some interval (ω', ω'') and let $f'(\omega)$ be continuous at some point ω_0 of this interval. Then*

$$\frac{\partial f(r, \omega)}{\partial \omega} \to f'(\omega_0),$$

where $M(r, \omega) \to P(1, \omega_0)$ along any path, provided it remains within a unit circle.

We have from (55.1)

$$\frac{\partial f(r, \omega)}{\partial \omega} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \frac{\partial P(r, t - \omega)}{\partial \omega} dt. \tag{56.1}$$

Since

$$\frac{\partial P(r, u)}{\partial u} = \frac{-(1 - r^2)\, 2r \sin u}{[1 - 2r \cos u + r^2]^2}, \tag{56.2}$$

then $\partial P(r, u)/\partial u$ is an odd function, negative or equal to zero in $[0, \pi]$, whilst for any $\delta > 0$ we have

$$\max_{\delta \leqslant |u| \leqslant \pi} \left| \frac{\partial P(r, u)}{\partial u} \right| \leqslant \frac{2(1 - r^2)}{[1 - 2r \cos\delta + r^2]^2} \to 0 \quad \text{as} \quad r \to 1. \tag{56.3}$$

We choose δ so that $(\omega_0 - \delta, \omega_0 + \delta)$ lies within (ω', ω'') and divide the integral (56.1) into two for the interval $(\omega_0 - \delta, \omega_0 + \delta)$ and for the remaining part of the

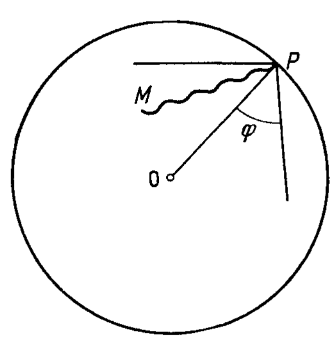

FIG. 12

circle. In the second integral for any $\varepsilon > 0$, provided M becomes sufficiently close to P, the modulus of the factor $\partial P(r, t - \omega)/\partial \omega$ becomes less than ε by virtue of (56.3), which means that the whole integral will not exceed $\varepsilon \int_{-\pi}^{\pi} |f(t)|\, dt$. As regards the first integral, integrating by parts, we have

$$\frac{1}{\pi}\int_{\omega_0-\delta}^{\omega_0+\delta} f(t)\frac{\partial P(r,t-\omega)}{\partial\omega}\,dt = -\frac{1}{\pi}\int_{\omega_0-\delta}^{\omega_0+\delta} f(t)\frac{\partial P(r,t-\omega)}{\partial t}\,dt$$

$$= -\frac{1}{\pi}[f(t)\,P(r,t-\omega)]\Big|_{\omega_0-\delta}^{\omega_0+\delta}$$

$$+\frac{1}{\pi}\int_{\omega_0-\delta}^{\omega_0+\delta} f'(t)\,P(r,t-\omega)\,dt.$$

Here the integrated term tends to zero when $M \to P$, because the Poisson kernel tends to 0, and $f(t)$ is bounded as far as the integral is concerned, so that it is possible to consider it to be the Poisson integral of the function equal to $f'(t)$ in $(\omega_0 - \delta, \omega_0 + \delta)$ and zero in the remaining part of the circle; this function, by hypothesis, is continuous at $\omega = \omega_0$, and therefore on the basis of the preceding results, this integral tends to $f'(\omega_0)$, no matter how M tends to P.

Thus our assertion concerning $\partial f(r,\omega)/\partial\omega$ is true and Lemma 1 is proved.

We shall now prove Theorem 1. First, we refute the hypothesis that $f'(\omega)$ is continuous at ω_0 and confine ourselves to the fact that it exists and is finite; then we will consider movement along non-tangential paths.

For simplicity of argument, we will suppose that $\omega_0 = 0$ and $f(0) = f'(0) = 0$ (this does not decrease the generality, as it is possible to consider instead of $f(\omega)$ the function $f_1(\omega) = f(\omega) - f(0) - \omega f'(0)$ and to study the behaviour of the Poisson integral for it).

Thus, we should prove that if $f(0) = f'(0) = 0$, then

$$\frac{\partial f(r,\omega)}{\partial\omega} \to 0, \tag{56.4}$$

if $M(r,\omega) \to P(1,0)$ by any tangential path.

First we note that because of our conditions we have $\lim\limits_{t\to 0}(f(t)/t) = 0$, and therefore for any $\varepsilon > 0$ it is possible to find $\delta > 0$ such that

$$\left|\frac{f(t)}{t}\right| < \varepsilon \quad \text{at} \quad |t| \leqslant \delta. \tag{56.5}$$

For the remainder it is convenient to take $\delta < \pi/2$.

Let $\Psi(t) = 0$ in $(-\delta, \delta)$, $\Psi(t) = f(t)$ in $\delta \leqslant |t| \leqslant \pi$ and $\Psi(t + 2\pi) = \Psi(t)$. It is clear then that denoting its Poisson integral by $\Psi(r,\omega)$, we have

$$\frac{\partial\Psi(r,\omega)}{\partial\omega} = \frac{1}{\pi}\int_{\delta\leqslant|t|\leqslant\pi} f(t)\frac{\partial P(r,t-\omega)}{\partial\omega}\,dt. \tag{56.6}$$

On the other hand, since $\Psi(t)$ satisfies the conditions of Lemma 1 in $(-\delta, \delta)$ and $\Psi'(0) = 0$, then $\partial\Psi(r,\omega)/\partial\omega \to 0$, when $M(r,\omega) \to P(1,0)$ along any path. Hence it follows that the integral on the right-hand side of (56.6) tends to zero, and therefore it follows from (56.1) that (56.4) will be true, if we prove that the integral

$$I = \frac{1}{\pi}\int_{-\delta}^{\delta} f(t)\frac{\partial P(r, t-\omega)}{\partial \omega}\,dt$$

can be made less than $C\varepsilon$ where C is a constant. But by virtue of (56.5) we have

$$|I| < \frac{\varepsilon}{\pi}\int_{-\delta}^{\delta}\left| t\frac{\partial P(r, t-\omega)}{\partial \omega}\right| dt.$$

We now prove that the expression

$$Q = t\frac{\partial P(r, t-\omega)}{\partial \omega} \tag{56.7}$$

remains bounded in $-\delta \leqslant t \leqslant \delta$, when the point $M(r, \omega) \to P(1, 0)$ along any non-tangential path.

In order to prove this, we remark first that on the basis of (56.2)

$$|Q| = |t|\frac{|2r\sin(t-\omega)(1-r^2)|}{|e^{it} - re^{i\omega}|^2} \leqslant \frac{2|t|\,|\sin(t-\omega)|}{|e^{it} - re^{i\omega}|^2}.$$

Since

$$|e^{it} - re^{i\omega}| = |e^{i(t-\omega)} - r| \leqslant |\sin(t-\omega)|,$$

because the modulus of a complex quantity is not less than the modulus of its imaginary part, then

$$|Q| \leqslant \frac{2|t|}{|e^{it} - re^{i\omega}|}.$$

Moreover, we note that $\delta \leqslant \pi/2$ and therefore $|t| \leqslant (\pi/2)|\sin t|$, whence

$$|Q| \leqslant \pi\frac{|\sin t|}{|e^{it} - re^{i\omega}|}. \tag{56.8}$$

We can confine ourselves to considering the case $-\pi/2 \leqslant \omega \leqslant \pi/2$, because $M(r, \omega) \to P(1, 0)$. Figure 13 holds for $\omega > 0$ but the case $\omega < 0$ is treated in an exactly similar manner.

Since we are concerned with non-tangential paths, there exists an angle KPK' with a vertex at P and the bisector OP such that the point M as it approaches P cannot go outside this angle. Letting $\alpha = KPy'$ where Py' is a line, passing through P, parallel to the axis Oy, we see that the vector PM forms with the positive direction of the abscissa axis an angle φ, where $\varphi \geqslant \pi/2 + \alpha$ (if $\omega < 0$, then we will have $\varphi \leqslant 3\pi/2 - \alpha$) whence it is clear that $re^{i\omega} = 1 + \varrho e^{i\varphi}$, where ϱ is the length of the vector MP and $\pi/2 + \alpha \leqslant \varphi \leqslant 3\pi/2 - \alpha$, i.e. $\alpha \leqslant \varphi - \pi/2 \leqslant \pi - \alpha$.

Thus

$$\varrho e^{i\varphi} = \varrho e^{i\frac{\pi}{2}} e^{i\left(\varphi - \frac{\pi}{2}\right)} = i\varrho e^{i\left(\varphi - \frac{\pi}{2}\right)} = i\varrho e^{i\Psi},$$

where $\alpha \leqslant \Psi \leqslant \pi - \alpha$. Therefore

$$\left|e^{it} - re^{i\omega}\right| = \left|e^{it} - 1 - i\varrho e^{i\Psi}\right| = \left|\frac{e^{it}-1}{i}e^{-i\Psi} - \varrho\right|$$

$$= \left|\frac{e^{i\frac{t}{2}} - e^{-i\frac{t}{2}}}{2i} 2e^{i\left(\frac{t}{2}-\Psi\right)} - \varrho\right| = \left|2\sin\frac{t}{2}e^{i\left(\frac{t}{2}-\Psi\right)} - \varrho\right|$$

$$\geqslant 2\left|\sin\frac{t}{2}\sin\left(\frac{t}{2} - \Psi\right)\right|. \tag{56.9}$$

(Here we again use the fact that the modulus of a complex quantity is not less than the modulus of its imaginary part.)

Now from (56.8) and (56.9) we conclude

$$|Q| \leqslant \pi \frac{|\sin t|}{2\left|\sin\frac{t}{2}\right|\left|\sin\left(\frac{t}{2} - \Psi\right)\right|} \leqslant \frac{\pi}{\left|\sin\left(\frac{t}{2} - \Psi\right)\right|}.$$

If $|t| < \alpha$, then $|\sin(t/2 - \Psi)| \geqslant \sin(\alpha/2)$ and then

$$|Q| \leqslant \frac{\pi}{\sin\frac{\alpha}{2}},$$

i.e. Q is bounded. If $\delta \geqslant |t| \geqslant \alpha$, then for $M(r, \omega) \to P(1, 0)$ the denominator in (56.8) is bounded below, which means that $|Q|$ is again bounded. This concludes the proof of Theorem 1.

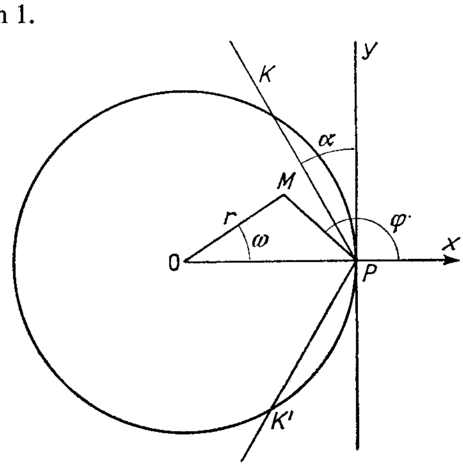

FIG. 13

Using the proved Theorem 1, we can now obtain a result referring to the behaviour of the Poisson integral for any summable function $f(x)$. We will prove the following theorem, also due to Fatou:

THEOREM 2. *At any point* ω_0, *where* $f(\omega)$ *is the derivative of its indefinite integral, the Poisson integral* $f(\omega, r) \to f(\omega_0)$, *if the point* $M(r, \omega)$ *tends to the point* $P(1, \omega_0)$ *along any non-tangential path.*

In particular it follows that *the Fourier series of any summable function is summable by Poisson's method to this function almost everywhere.*†

In order to prove this we will suppose that

$$F(\omega) = \int_{-\pi}^{\omega} f(t)\, dt.$$

We have, integrating by parts,

$$f(r, \omega) = \frac{1}{\pi}[F(t)\, P(r, t - \omega)] \Big|_{-\pi}^{\pi} - \frac{1}{\pi} \int_{-\pi}^{\pi} F(t) \frac{\partial}{\partial t}[P(r, t - \omega)]\, dt.$$

The integrated term tends to zero when $M \to P(1, \omega_0)$ provided $\omega \neq -\pi$ and $\omega \neq \pi$. As regards the integral, it is possible to rewrite it in the form

$$\frac{\partial}{\partial \omega}\left\{\frac{1}{\pi}\int_{-\pi}^{\pi} F(t)\, P(r, t - \omega)\, dt\right\} = \frac{\partial}{\partial \omega} F(r, \omega), \tag{56.10}$$

and therefore, only on the basis of the result just obtained, if $M(r, \omega) \to P(1, \omega_0)$ along a non-tangential path, the expression (56.10) tends to $F'(\omega_0)$ everywhere, where $F'(x)$ exists and is finite. Consequently, at any point where $f(\omega_0) = F'(\omega_0)$ we have $f(r, \omega) \to f(\omega_0)$ and this is what was required to be proved.

Since from the theory of the Lebesgue integral it is known that the equality $F'(\omega) = f(\omega)$ holds almost everywhere, then it follows in particular that for almost all values of ω

$$f(r, \omega') \to f(\omega),$$

where $M(r, \omega') \to P(1, \omega)$ along any non-tangential path. This occurs even more so, when $M(r, \omega) \to P(1, \omega)$ as $r \to 1$, whence it is evident that the theorem of § 53 is a corollary of Fatou's theorem.

We will now look at the role played by the Poisson integral in solving the celebrated Dirichlet problem.

§ 57. The Dirichlet problem

This problem was set by Dirichlet in the following form: Given a closed contour and a function $f(x)$, continuous on it, it is required to find a harmonic†† function inside this contour tending to given values on the contour when the point tends by any method from inside to the periphery.

† Moreover, it is summable almost everywhere to $f(x)$ by method A^* (see the definition of A^* in § 7 of the Introductory Material).

†† That is, it satisfies Laplace's equation

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0.$$

We will discuss the particular case when the contour under consideration is a circle of unit radius with centre at the origin. If we denote by x and y the Cartesian co-ordinates of the point $M(r, \omega)$ then we have

$$F(x, y) = f(r, \omega) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega + b_n \sin n\omega) r^n,$$

where a_n and b_n are the Fourier coefficients for $f(x)$ and therefore $F(x, y)$ is the real part of the analytic function inside the circle of unit radius, defined by the power series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n - ib_n) z^n, \quad z = re^{i\omega}.$$

But it is known that the real (and imaginary) part of any analytic function is a harmonic function, that is, it follows from the theorem of § 55 that the function $F(x, y)$ gives the solution of the Dirichlet problem for a circle.

If the Dirichlet problem is extended by not requiring the values of the function given on the boundary to define a continuous function, but permitting the point to tend from inside to the periphery only along non-tangential paths, then $F(x, y)$ tends to $f(\omega)$ almost everywhere and thus gives the solution of the generalized Dirichlet problem.

§ 58. Summation by Poisson's method of a differentiated Fourier series

Let

$$\sigma(F) = \frac{A_0}{2} + \sum (A_0 \cos nx + B_n \sin nx). \tag{58.1}$$

We know that the series

$$\sum n(B_n \cos nx - A_n \sin nx), \tag{58.2}$$

obtained by differentiating (58.1) should not be a Fourier series, since its coefficients

$$a_n = nB_n \quad \text{and} \quad b_n = -nA_n$$

should not even tend to zero. Therefore, the preceding theorems cannot be applied to series (58.2). But instead we have the following:

FATOU'S THEOREM. *If at some point x the function $F(x)$ has a symmetrical derivative equal to the value l, then by differentiating the series $\sigma(F)$, we obtain a series which is summable at the point x to the value l by Poisson's method.*

Since $\lim_{h\to 0} [F(x + h) - F(x - h)]/2h$, if this limit exists, is the symmetrical derivative, then by the condition of the theorem

$$\lim_{h\to 0} \frac{F(x + h) - F(x - h)}{2h} = l. \tag{58.3}$$

Supposing, as always

$$F(r, x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n \cos nx + B_n \sin nx) r^n,$$

we can write

$$\frac{\partial F(r, x)}{\partial x} = \sum_{n=1}^{\infty} (B_n \cos nx - A_n \sin nx) r^n. \tag{58.4}$$

Here term-by-term differentiation is valid, since at $r < 1$ series (58.4) converges uniformly relative to x. It is necessary for us to prove that

$$\frac{\partial F(r, x)}{\partial x} \to l \quad \text{as} \quad r \to 1.$$

But

$$\frac{\partial F(r, x)}{\partial x} = \frac{1}{\pi} \int_{-\pi}^{\pi} F(t) \frac{\partial P(r, t - x)}{\partial x} dt = -\frac{1}{\pi} \int_{-\pi}^{\pi} F(t) \frac{\partial P(r, t - x)}{\partial t} dt, \tag{58.5}$$

and since $\partial P(r, u)/\partial u$ is an odd function (see (56.2)), then

$$\frac{\partial F(r, x)}{\partial x} = -\frac{1}{\pi} \int_{-\pi}^{\pi} F(x + u) \frac{\partial P(r, u)}{\partial u} du$$

$$= -\frac{1}{\pi} \int_{0}^{\pi} [F(x + u) - F(x - u)] \frac{\partial P(r, u)}{\partial u} du.$$

By virtue of (56.3) for any $\varepsilon > 0$ and $\delta > 0$ it is possible to choose $r_0 < 1$ such that $|\partial P(r, u)/\partial u| < \varepsilon$ for $\delta \leqslant u \leqslant \pi$ and $r_0 \leqslant r < 1$. Therefore

$$\frac{\partial F(r, x)}{\partial x} = -\frac{1}{\pi} \int_{0}^{\delta} [F(x + u) - F(x - u)] \frac{\partial P(r, u)}{\partial u} du + I_1, \tag{58.6}$$

where

$$|I_1| \leqslant \frac{2\varepsilon}{\pi} \int_{-\pi}^{\pi} |F(t)| \, dt < C\varepsilon, \tag{58.7}$$

where C is a constant. From (58.3) it is possible to suppose that the number δ is so small that

$$\left| \frac{F(x + u) - F(x - u)}{2u} - l \right| < \varepsilon. \tag{58.8}$$

Then from (58.6), (58.7) and (58.8)

$$\frac{\partial F(r, x)}{\partial x} = -\frac{l}{\pi} \int_{0}^{\delta} \frac{F(x + u) - F(x - u)}{2u} 2u \frac{\partial P(r, u)}{\partial u} du + I_1$$

$$= I_1 - \frac{1}{\pi} \int_{0}^{\delta} \left[\frac{F(x + u) - F(x - u)}{2u} - l \right] 2u \frac{\partial P(r, u)}{\partial u} du$$

$$- \frac{l}{\pi} \int_{0}^{\delta} 2u \frac{\partial P(r, u)}{\partial u} du = I_1 + I_2 + I_3. \tag{58.9}$$

Due to (58.8) we have

$$|I_2| < \frac{2\varepsilon}{\pi} \int_0^\delta \left| u \frac{\partial P(r, u)}{\partial u} \right| du < C_1 \varepsilon, \tag{58.10}$$

where C_1 is a constant. Indeed, from (56.2) we see that

$$\left| u \frac{\partial P(r, u)}{\partial u} \right| \leqslant \left| \frac{2u \sin u}{\sin^2 u} \right| \leqslant \pi, \quad \text{for} \quad 0 \leqslant u \leqslant \frac{\pi}{2}.$$

For I_3, integrating by parts, we find that

$$I_3 = -\frac{2l}{\pi} \int_0^\delta u \frac{\partial P(r, u)}{\partial u} du = -\frac{2l}{\pi} \delta P(r, \delta) + \frac{2l}{\pi} \int_0^\delta P(r, u)\, du \to l. \tag{58.11}$$

because $P(r, \delta) \to 0$ and from formula (55.4).

Now from (58.7), (58.10) and (58.11) we obtain

$$\frac{\partial F(r, x)}{\partial x} \to l \quad \text{as} \quad r \to 1,$$

and the theorem is proved.

Note. Since the presence of the normal derivative at some point guarantees the existence of a symmetrical derivative at that point and their equality, then from this it follows in particular that:

If at some point x the derivative $F'(x)$ exists and is finite, then $\partial F(r, x)/\partial x \to F'(x)$ as $r \to 1$, i.e. when $F(x)$ has a finite derivative, the differentiated Fourier series is summable to this derivative by Poisson's method.

In § 56 we have essentially already obtained this result (only it is formulated in different terms). Now we will see that the requirement of the existence of $F'(x)$ can be replaced by the weaker requirement of the existence of a symmetrical derivative. But whereas in the theorem of this section $M(r, x_0) \to P(1, x_0)$ along the radial path, in § 56 it was proved that $M(r, x) \to P(1, x_0)$ along any non-tangential path.

§ 59†. Poisson–Stieltjes integral

The Poisson–Stieltjes integral is the name given to the expression

$$u(re^{i\omega}) = \frac{1}{\pi} \int_{-\pi}^{\pi} P(r, t - \omega)\, d\,\Psi(t),$$

where $\Psi(t)$ is some function of bounded variation in $[-\pi, \pi]$. Integrating by parts, we obtain

$$u(re^{i\omega}) = \frac{1}{\pi} P(r, t - \omega)\, \Psi(t) \Big|_{-\pi}^{\pi} - \frac{1}{\pi} \int_{-\pi}^{\pi} \Psi(t) \frac{\partial}{\partial t} P(r, t - \omega)\, dt.$$

† This section can be omitted at a first reading.

If $\omega \neq \pm\pi$, then the integrated term as $r \to 1$ tends to zero. As regards the integral, from Theorem 1, § 56 it should tend to $\Psi'(\omega_0)$ at any point ω_0, where $\Psi'(\omega)$ exists and is finite, only if the point $M(re^{i\omega})$ tends to the point $P(e^{i\omega_0})$ along any non-tangential path.

In particular

$$u(re^{i\omega}) \to \Psi'(\omega) \quad \text{as} \quad r \to 1,$$

if $\Psi'(\omega)$ exists and is finite.

Hence as a corollary we obtain: *the Fourier–Stieltjes series is summable by the Abel–Poisson method almost everywhere.*

Later we will find it useful to prove that if $\omega \neq \pm\pi$ and $\Psi'(\omega) = +\infty$, then we have

$$u(re^{i\omega}) \to +\infty \quad \text{as} \quad r \to 1.$$

In order to prove this, from what has been said concerning the integrated term, it is sufficient to prove that

$$I = -\frac{1}{\pi}\int_{-\pi}^{\pi} \Psi(t)\frac{\partial}{\partial t}P(r, t-\omega)\,dt \to +\infty \quad \text{as} \quad r \to 1.$$

It is just the same kind of integral as (58.5), therefore we see immediately that for any $\varepsilon > 0$

$$I = -\frac{1}{\pi}\int_0^{\delta} [\Psi(\omega+u) - \Psi(\omega-u)]\frac{\partial P(r,u)}{\partial u}\,du + I_1 = I_1 + I_2,$$

where $|I_1| < \varepsilon$, if δ is fixed and r is taken sufficiently close to 1. Now we represent I_2 in the form

$$I_2 = -\frac{1}{\pi}\int_0^{\delta} [\Psi(\omega+u) - \Psi(\omega)]\frac{\partial P(r,u)}{\partial u}\,du$$

$$-\frac{1}{\pi}\int_0^{\delta} [\Psi(\omega) - \Psi(\omega-u)]\frac{\partial P(r,u)}{\partial u}\,du = I_3 + I_4. \tag{59.1}$$

We will show that $I_3 \to +\infty$ and $I_4 \to +\infty$. The proof for both integrals is completely identical. We will carry it through for I_3.

Since $\Psi'(x) = +\infty$, we can, if A is given, suppose that δ is so small that

$$\Psi(\omega+u) - \Psi(\omega) > Au \quad \text{for} \quad 0 \leqslant u \leqslant \delta.$$

We have

$$I_3 = -\frac{1}{\pi}\int_0^{\delta} \frac{\Psi(\omega+u) - \Psi(u)}{u}\left[u\frac{\partial P(r,u)}{\partial u}\right]du,$$

but $-u\,\partial P(r,u)/\partial u \geqslant 0$ (see (56.2)), therefore

$$I_3 > A\frac{1}{\pi}\int_0^{\delta}\left[-u\frac{\partial P(r,u)}{\partial u}\right]du$$

and we have seen (see (58.11)) that

$$\frac{2}{\pi}\int_0^\delta \left[-u\frac{\partial P(r,u)}{\partial u}\right] du \to 1 \quad \text{as} \quad r \to +1,$$

whence it follows that as $r \to 1$ it is possible to make $I_3 > A/2$ where A is previously given and the proof is concluded.

Note. That the Fourier–Stieltjes series or the series obtained after differentiation of the Fourier series for a function of bounded variation (see § 23) cannot be a Fourier series is evident from this simple example: the series

$$\sum_{n=1}^{\infty}\frac{\sin nx}{n},$$

as we know (see § 41) is the Fourier series of a function monotonic in $[0, 2\pi]$; however after its differentiation we obtain the series

$$\sum_{n=1}^{\infty}\cos nx,$$

which is not a Fourier series because its coefficients do not tend to zero.

§ 60. Fejér and Poisson sums for different classes of functions

We will now prove a number of theorems which will show that it is possible to judge the properties of a function by studying the sequence of its Fejér or Poisson sums.

THEOREM 1. *In order for the trigonometric series to be a Fourier series for a continuous function, it is necessary and sufficient for the sequence of its Fejér sums $\{\sigma_n(x)\}$ to converge uniformly.*

The necessity of the condition is given simply by Fejér's theorem (see § 47). To prove its sufficiency, we note that if the given trigonometric series is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

then

$$\sigma_n(x) = \sum_{k=0}^{n}\left(1 - \frac{k}{n+1}\right)(a_k \cos kx + b_k \sin kx), \tag{60.1}$$

and therefore for $k \leqslant n$

$$\left.\begin{aligned}\left(1 - \frac{k}{n+1}\right) a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi}\sigma_n(t)\cos kt\, dt,\\ \left(1 - \frac{k}{n+1}\right) b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi}\sigma_n(t)\sin kt\, dt.\end{aligned}\right\} \tag{60.2}$$

If the sequence $\sigma_n(x)$ converges uniformly, then supposing $f(x) = \lim_{n\to\infty} \sigma_n(x)$, we see that $f(x)$ is continuous. As $n \to \infty$ from equations (60.2) by passing to the limit we obtain

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\,dt \quad (k = 0, 1, \ldots),$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt\,dt \quad (k = 1, 2, \ldots),$$

and this is what was required to be proved.

THEOREM 2. *For the trigonometric series to be a Fourier series for a bounded function, it is necessary and sufficient for a constant K to be found for which*

$$|\sigma_n(x)| \leqslant K \quad (n = 1, 2, \ldots;\ 0 \leqslant x \leqslant 2\pi).$$

The necessity of this condition was proved in § 48. To prove its sufficiency we note that if it is satisfied, then

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sigma_n^2(x)\,dx \leqslant 2K^2.$$

But due to Parseval's equality we obtain from (60.1)

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sigma_n^2(x)\,dx = \frac{a_0^2}{2} + \sum_{k=1}^{n}\left(1 - \frac{k}{n+1}\right)^2 (a_k^2 + b_k^2).$$

From this it follows that if m is any integer, $m \leqslant n$, then

$$\frac{a_0^2}{2} + \sum_{k=1}^{m}\left(1 - \frac{k}{n+1}\right)^2 (a_k^2 + b_k^2) \leqslant 2K^2.$$

Letting $n \to \infty$ and keeping m constant, we conclude from this that

$$\frac{a_0^2}{2} + \sum_{k=1}^{m} (a_k^2 + b_k^2) \leqslant 2K^2$$

and since m is any number, the series $\sum(a_k^2 + b_k^2) < +\infty$.

This means that the trigonometric series under consideration is a Fourier series of some function $f(x) \in L^2$. But since $\sigma_n(x) \to f(x)$ almost everywhere, then from $|\sigma_n(x)| \leqslant K$ it follows that $|f(x)| \leqslant K$ and the theorem is proved.

THEOREM 3. *For the trigonometric series to be a Fourier series for $f(x) \in L^p (p > 1)$, it is necessary and sufficient that*

$$\|\sigma_n(x)\|_{Lp} \leqslant K \quad (n = 1, 2, \ldots), \tag{60.3}$$

where K is a constant.

To prove the necessity we note that

$$\sigma_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) K_n(t-x)\,dt.$$

Therefore, noting that $(1/\pi)\int_{-\pi}^{\pi} K_n(u)\,du = 1$ and that $K_n(u) \geqslant 0$ and applying the lemma proved in § 9 of the Introductory Material, we immediately find

$$\|\sigma_n(x)\|_{L^p} \leqslant \|f(x)\|_{L^p} \tag{60.4}$$

and since the right-hand side of (60.4) does not vary with n, the proof is concluded.

To prove the sufficiency we will consider the functions

$$F_n(x) = \int_0^x \sigma_n(t)\,dt \tag{60.5}$$

and prove that they are uniformly absolutely continuous, i.e. for any ε there exists δ such that for any system of non-overlapping intervals (a_i, b_i) with a sum $\sum (b_i - a_i) < \delta$ we have

$$\sum |F_n(b_i) - F_n(a_i)| < \varepsilon. \tag{60.6}$$

Indeed, denoting by S this system of intervals, because of (60.3), we have

$$\sum |F_n(b_i) - F_n(a_i)| \leqslant \sum \int_{a_i}^{b_i} |\sigma_n(t)|\,dt$$
$$= \int_S |\sigma_n(t)|\,dt \leqslant \left(\int_S |\sigma_n(t)|^p dt\right)^{1/p} \left(\int_S 1^q\,dt\right)^{1/q} \leqslant \delta^{1/q}\|\sigma_n\|_{Lp} \leqslant \delta^{1/q} K < \varepsilon,$$

if δ is sufficiently small.

Arguing this, we see that the complete variations of these functions are all bounded. Therefore, from Helly's theorem (see Introductory Material, § 17) it is possible to extract from them the sub-sequence $F_{n_j}(x)$ which converges at every point to some function $F(x)$; according to Helly's theorem it should be of bounded variation, but from the uniform absolute continuity of the function $F_n(x)$ it immediately follows that it is absolutely continuous.

In fact, if in formula (60.6) instead of n we write n_j and pass to the limit as $j \to \infty$, then we obtain

$$\sum |F(b_i) - F(a_i)| < \varepsilon.$$

Let us prove now that the series under consideration is $\sigma(f)$ where $f(x) = F'(x)$.

Indeed, we have

$$\int_0^{2\pi} \sigma_n(t)\cos kt\,dt = F_n(t)\cos kt\Big|_0^{2\pi} + k\int_0^{2\pi} F_n(t)\sin kt\,dt = F_n(2\pi) + k\int_0^{2\pi} F_n(t)\sin kt\,dt$$

and

$$\int_0^{2\pi} \sigma_n(t)\sin kt\,dt = -k\int_0^{2\pi} F_n(t)\cos kt\,dt.$$

Letting $n \to \infty$ for the sequence n_j, for which $F_{n_j}(x) \to F(x)$, we obtain from formula (60.2)

$$a_k = \frac{1}{\pi} F(2\pi) + \frac{k}{\pi} \int_0^{2\pi} F(t) \sin kt \, dt, \quad b_k = -\frac{k}{\pi} \int_0^{2\pi} F(t) \cos kt \, dt$$

(passage to the limit under the integral sign is valid here due to Lebesgue's theorem (see Introductory Material, § 14)).

After integration by parts of the last two integrals we conclude from this that

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos kt \, dt, \quad b_k = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin kt \, dt,$$

and this is what was required to be proved.

It remains to prove that $f(x) \in L^p$. But for this it is sufficient to note that $\sigma_n(x) \to f(x)$ almost everywhere, then, using the inequality (60.3) and Fatou's lemma (see Introductory Material, § 14), we immediately obtain $\|f(x)\|_{L^p} \leqslant K$.

Corollary. *If $f(x) \in L^p$, $p > 1$, then*

$$\int_0^{2\pi} |f(x) - \sigma_n(x)|^p dx \to 0 \quad \text{as} \quad n \to \infty. \tag{60.7}$$

We already know (see (60.4)) that if $f(x) \in L^p$, then

$$\|\sigma_n(x)\|_{L^p} \leqslant \|f(x)\|_{L^p}.$$

Let $\varepsilon > 0$ be given. It is possible to find (see § 28) a trigonometric polynomial $T(x)$ such that

$$\|f(x) - T(x)\|_{L^p} < \varepsilon. \tag{60.8}$$

Consequently, for any n

$$\|\sigma_n(x, f - T)\|_{L^p} < \varepsilon,$$

i.e.

$$\|\sigma_n(x, f) - \sigma_n(x, T)\|_{L^p} < \varepsilon. \tag{60.9}$$

But since $T(x)$ is a trigonometric polynomial, then the continuous function $\sigma_n(x, T)$ tends to $T(x)$ uniformly, and even more so

$$\|\sigma_n(x, T) - T(x)\|_{L^p} < \varepsilon \tag{60.10}$$

provided n becomes sufficiently large. Therefore, from (60.8), (60.9) and (60.10) we have

$$\|f(x) - \sigma_n(x, f)\|_{L^p} \leqslant \|f(x) - T(x)\|_{L^p} + \|T(x) - \sigma_n(x, T)\|_{L^p}$$
$$+ \|\sigma_n(x, T) - \sigma_n(x, f)\|_{L^p} \leqslant 3\varepsilon,$$

if n is sufficiently large and thus (60.7) is proved.

Below, in the proof of Theorem 4, we shall see that this assertion holds too for $p = 1$, i.e. if $f(x) \in L$, then

$$\int_0^{2\pi} |f(x) - \sigma_n(x)| \, dx \to 0 \quad \text{as} \quad n \to \infty.$$

However, Theorem 3 holds only for $p > 1$. In fact, if $p = 1$, i.e. if

$$\int_0^{2\pi} |\sigma_n(x)|\, dx \leqslant K,$$

then we cannot assert that the series under consideration is a Fourier series (see the note to Theorem 5 below). The case of a Fourier series is considered in the following theorem:

THEOREM 4. *In order for a trigonometric series to be a Fourier series, it is necessary and sufficient that*

$$\int_0^{2\pi} |\sigma_m(x) - \sigma_n(x)|\, dx \to 0 \quad \textit{as} \quad m \to \infty \quad \textit{and} \quad n \to \infty.$$

We know (see § 47) that

$$\sigma_n(x) - f(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} [f(x+t) - f(x)] K_n(t)\, dt.$$

Therefore, supposing that

$$\Psi(t) = \int_{-\pi}^{\pi} |f(x+t) - f(x)|\, dx,$$

we have

$$\int_{-\pi}^{\pi} |\sigma_n(x) - f(x)|\, dx \leqslant \int_{-\pi}^{\pi} \left\{ \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x+t) - f(x)|\, K_n(t)\, dt \right\} dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \Psi(t) K_n(t)\, dt. \tag{60.11}$$

If we denote by $\sigma_n^*(x)$ the Fejér sum for $\sigma(\Psi)$, then

$$\sigma_n^*(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} \Psi(t+x) K_n(t)\, dt,$$

and therefore, from (60.11)

$$\int_{-\pi}^{\pi} |\sigma_n(x) - f(x)|\, dx \leqslant \sigma_n^*(0).$$

But since $\Psi(t)$ is continuous and $\Psi(0) = 0$, then $\sigma_n^*(0) \to 0$ as $n \to \infty$, which means that

$$\int_{-\pi}^{\pi} |\sigma_n(x) - f(x)|\, dx \to 0. \tag{60.12}$$

From this we obtain

$$\int_{-\pi}^{\pi} |\sigma_n(x) - \sigma_m(x)|\, dx \leqslant \int_{-\pi}^{\pi} |\sigma_n(x) - f(x)|\, dx + \int_{-\pi}^{\pi} |f(x) - \sigma_m(x)|\, dx \to 0$$

and the necessity of our condition is proved.

To prove its sufficiency we note that from

$$\int_0^{2\pi} |\sigma_n(x) - \sigma_m(x)|\, dx \to 0 \quad \text{as} \quad m \to \infty,\ n \to \infty$$

the existence of the constant K, for which

$$\int_0^{2\pi} |\sigma_n(x)|\, dx \leqslant K \quad (n = 1, 2, \ldots).$$

follows. Supposing, as in Theorem 3,

$$F_n(x) = \int_0^x \sigma_n(t)\, dt.$$

then we will prove in the same way that the sequence of functions $F_n(x)$ is uniformly absolutely continuous. Here, using the notation of Theorem 3, we have

$$\sum |F_n(b_i) - F_n(a_i)| \leqslant \int_S |\sigma_n(t)|\, dt. \tag{60.13}$$

But

$$\int_S |\sigma_n(t)|\, dt \leqslant \int_S |\sigma_n(t) - \sigma_k(t)|\, dt + \int_S |\sigma_k(t)|\, dt$$

$$\leqslant \int_0^{2\pi} |\sigma_n(t) - \sigma_k(t)|\, dt + \int_S |\sigma_k(t)|\, dt. \tag{60.14}$$

Let $\varepsilon > 0$ be given. Due to the condition of the theorem it is possible to take k so large that

$$\int_0^{2\pi} |\sigma_n(t) - \sigma_k(t)|\, dt < \frac{\varepsilon}{2} \quad \text{for} \quad n \geqslant k. \tag{60.15}$$

We will now fix k; then, taking δ sufficiently small, it can be proved that $\int_S |\sigma_p(t)|\, dt < \varepsilon/2$ at $p \leqslant k$ provided $mS < \delta$. But if this is so, then from (60.14) and (60.15)

$$\int_S |\sigma_n(t)|\, dt < \varepsilon$$

and consequently from (60.13)

$$\sum |F_n(b_i) - F_n(a_i)| < \varepsilon \quad \text{at} \quad \sum (b_i - a_i) < \delta.$$

Now as in Theorem 3 we see that it is possible to remove from the sequence $\{F_n(x)\}$ a sub-sequence converging to some $F(x)$ which should be absolutely continuous and moreover, the series under consideration is a Fourier series of $F'(x)$.

The theorem is proved.

Note. In the process of this proof we have established that for any $f(x) \in L$ we have

$$\int_0^{2\pi} |f(x) - \sigma_n(x)|\, dx \to 0 \quad \text{as} \quad n \to \infty. \tag{60.16}$$

Finally, we shall prove yet another theorem.

THEOREM 5. *In order for the trigonometric series to be a Fourier–Stieltjes*† *series, it is necessary and sufficient that*

$$\int_{-\pi}^{\pi} |\sigma_n(x)|\, dx \leqslant K \quad (n = 1, 2, \ldots),$$

where K is a constant.

The necessity of the condition follows from the fact that for a Fourier–Stieltjes series we have

$$\sigma_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} K_n(t - x)\, dF(t) \tag{60.17}$$

(this formula is derived in just the same way as (47.2)). Therefore

$$|\sigma_n(x)| \leqslant \frac{1}{\pi}\int_{-\pi}^{\pi} K_n(t - x)\, |dF|,$$

where $|dF|$ is no different from $dV(t)$, if $V(t)$ is taken to be the complete variation of $F(x)$ in $0 \leqslant x \leqslant t$. Hence, by changing the order of integration, we obtain

$$\int_0^{2\pi} |\sigma_n(x)|\, dx \leqslant \int_0^{2\pi} \left\{\frac{1}{\pi}\int_0^{2\pi} K_n(t - x)\, |dF(t)|\right\} dx$$

$$= \int_0^{2\pi} |dF(t)| \frac{1}{\pi}\int_0^{2\pi} K_n(t - x)\, dx = \int_0^{2\pi} |dF| = V,$$

where V is the complete variation of $F(x)$ in $[0, 2\pi]$.

Thus, the necessity is proved.

In order to prove the sufficiency, we will again turn to considering the function $F_n(x)$ already considered in Theorems 3 and 4. It is true that we have not been able to prove that they are uniformly absolutely continuous but nevertheless they are of uniformly bounded variation, because

$$\sum |F_n(x_{i+1}) - F_n(x_i)| \leqslant \sum \int_{x_i}^{x_{i+1}} |\sigma_n(t)|\, dt = \int_0^{2\pi} |\sigma_n(t)|\, dt \leqslant K$$

for any division of the interval $[0, 2\pi]$ by the points x_i. Therefore, from Helly's first theorem (see Introductory Material, § 17) a subsequence n_j exists such that $F_{n_j}(x) \to F(x)$ for any x of $[0, 2\pi]$, where $F(x)$ is of bounded variation. It remains to prove that the given series is the Fourier–Stieltjes series of dF.

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† See § 23, point (9).

For this, as in the proof of Theorem 3, we have

$$\left(1 - \frac{k}{n+1}\right) a_k = \frac{1}{\pi} \int_0^{2\pi} \sigma_n(t) \cos kt \, dt = \frac{F_n(2\pi)}{\pi} + \frac{k}{\pi} \int_0^{2\pi} F_n(t) \sin kt \, dt$$

and then by integrating by parts we obtain

$$\left(1 - \frac{k}{n+1}\right) a_k = \frac{1}{\pi} \int_0^{2\pi} \cos kt \, dF_n(t).$$

Permitting n to tend to infinity for the sub-sequence n_j, we find

$$a_k = \frac{1}{\pi} \int_0^{2\pi} \cos kt \, dF(t)$$

and similarly for b_k (passage to the limit is valid from Helly's second theorem, Introductory Material, § 17).

The theorem is proved.

Note. We know (see § 59) that not every Fourier–Stieltjes series is a Fourier series. Thus, the condition

$$\int_{-\pi}^{\pi} |\sigma_n(x)| \, dx \leqslant K \quad (n = 1, 2, \ldots)$$

is not sufficient for the series to be a Fourier series and this shows that at $p = 1$ Theorem 3 no longer holds.

Taking into account that the Fourier–Stieltjes series is the result of differentiating the Fourier series for a function of bounded variation, we obtain as a corollary of Theorem 5 the following theorem:

THEOREM 6. *For the trigonometric series to be the Fourier series of a function of bounded variation, it is necessary and sufficient that*

$$\int_0^{2\pi} |\sigma_n'(x)| \, dx \leqslant K \quad (n = 1, 2, \ldots).$$

All the theorems that have been proved have referred to Fejér sums. If instead of them we consider Poisson sums, i.e.

$$f(r, x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) r^n$$

and note that

$$f(r, x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) P(r, t - x) \, dt,$$

where $P(r, u)$ is a Poisson kernel, then it is possible to prove completely analogous theorems; indeed, in this proof we used the whole time the expression $\sigma_n(x)$ in the form

$$\sigma_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)K_n(t-x)\,dt$$

and we based the proof only on the facts that $K_n(u) \geqslant 0$ and

$$\frac{1}{\pi}\int_{-\pi}^{\pi} K_n(u)\,du = 1.$$

But we also have $P(r, u) \geqslant 0$ and

$$\frac{1}{\pi}\int_{-\pi}^{\pi} P(r, u)\,du = 1.$$

Therefore the whole argument can be carried out word for word (the fact that $r \to 1$ for all values of r, not just for a sequence, does not play a part, since it would be possible to consider the sequence $r_k \to 1$ as $k \to \infty$ and to use the kernels $P(r_k, u)$ in the discussion).

Thus the following theorems are obtained.

THEOREM 1′. *In order for a trigonometric series to be a Fourier series of a continuous function, it is necessary and sufficient for its Poisson sums $f(r, x)$ to tend uniformly to a limit as $r \to 1$.*

THEOREM 2′. *In order for a trigonometric series to be a Fourier series of a continuous function, it is necessary and sufficient that a constant K exists, for which*

$$|f(r, x)| \leqslant K, \quad \begin{array}{l} 0 \leqslant r < 1, \\ 0 \leqslant x \leqslant 2\pi. \end{array}$$

THEOREM 3′. *In order for a trigonometric series to be a Fourier series for $f(x) \in L^p$ $(p > 1)$, it is necessary and sufficient for*

$$\|f(r, x)\|_{L^p} \leqslant K, \quad 0 \leqslant r < 1.$$

Moreover, if $f(x) \in L^p$ $(p > 1)$, then

$$\|f(r, x)\|_{L^p} \leqslant \|f(x)\|_{L^p}. \tag{60.18}$$

We also have

$$\int_0^{2\pi} |f(x) - f(r, x)|^p dx \to 0 \quad \textit{as} \quad r \to 1, \tag{60.19}$$

whilst this is true both for $p > 1$ and for $p = 1$.

THEOREM 4′. *In order for the trigonometric series to be a Fourier series, it is necessary and sufficient that*

$$\int_0^{2\pi} |f(r, x) - f(\varrho, x)|\,dx \to 0 \quad \textit{as} \quad r \to 1 \quad \textit{and} \quad \varrho \to 1.$$

For the case of a Fourier–Stieltjes series the argument is somewhat more complicated. We will not go through it, but will confine ourselves to formulating the theorem analogous to Theorem 5, namely:

THEOREM 5′. *For the trigonometric series to be a Fourier–Stieltjes series, it is necessary and sufficient that*

$$\int_0^{2\pi} |f(r, x)|\, dx \leqslant K, \quad 0 \leqslant r < 1.$$

We note that in Chapter VIII (§ 14 and § 20) instead of Fejér or Poisson sums of a Fourier series we shall study its partial sums $S_n(x)$ and for them we shall consider the question of the behaviour of $\|S_n\|_{L^p}$ and $\|f - S_n\|_{L^p}$ at $p \geqslant 1$.

§ 61. General trigonometric series. The Lusin–Denjoy theorem

Up until now we have studied Fourier series. Now we will consider trigonometric series of the same general type and prove a number of very simple but important theorems concerning them. We will begin by considering the question of when the trigonometric series converges absolutely in a set of positive measure. Here we have a theorem proved simultaneously and independently by Lusin[3] and Denjoy[2].

THE LUSIN–DENJOY THEOREM. *If the trigonometric series*

$$\frac{a_o}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{61.1}$$

converges absolutely in the set E, $mE > 0$, then

$$\sum (|a_n| + |b_n|) < +\infty.$$

Let us define $\varrho_n = \sqrt{a_n^2 + b_n^2}$, $(n = 1, 2, \ldots)$ and let

$$a_0 = 0, \quad \frac{a_2}{2} = \varrho_0, \quad a_n = \varrho_n \cos\alpha_n, \quad b_n = \varrho_n \sin\alpha_n \quad (n = 1, 2, \ldots).$$

Then the series (61.1) takes the form

$$\sum_{n=0}^{\infty} \varrho_n \cos(nx - \alpha_n). \tag{61.2}$$

Absolute convergence of the series (61.2) in E means that

$$\sum_{n=0}^{\infty} \varrho_n |\cos(nx - \alpha_n)| < +\infty \quad \text{for} \quad x \in E. \tag{61.3}$$

According to Yegorov's theorem, it is possible to find a perfect set $P \subset E, mP > 0$, in which the series (61.3) converges uniformly. Let $S(x)$ be its sum in P, then from the uniform convergence of (61.3)

$$\int_P S(x)\, dx = \sum_{n=0}^{\infty} \varrho_n \int_P |\cos(nx - \alpha_n)|\, dx.$$

But

$$\int_P |\cos(nx - \alpha_n)|\, dx \geqslant \int_P \cos^2(nx - \alpha_n)\, dx$$
$$= \frac{1}{2}\int_P [1 + \cos 2(nx - \alpha_n)]\, dx = \frac{1}{2} mP + \frac{1}{2}\int_P \cos 2(nx - \alpha_n) dx.$$

If $f(x)$ denotes a function equal to 1 in P and zero outside it, then

$$\int_P \cos 2(nx - \alpha_n)\, dx = \int_{-\pi}^{\pi} f(x)\cos 2(nx - \alpha_n)\, dx$$
$$= \cos 2\alpha_n \int_{-\pi}^{\pi} f(x)\cos 2nx\, dx + \sin 2\alpha_n \int_{-\pi}^{\pi} f(x)\sin 2nx\, dx, \quad (61.4)$$

and therefore

$$\int_P \cos 2(nx - \alpha_n)\, dx \to 0 \quad \text{as} \quad n \to \infty,$$

since the integrals on the right-hand side of (61.4) differ only by a bounded multiplier from the Fourier coefficients of $f(x)$.

From this it follows that

$$\int_P |\cos(nx - \alpha_n)|\, dx > \frac{1}{4} mP$$

where n is sufficiently large, which means that the convergence of series (61.3) implies the convergence of the series $\sum \varrho_n$, whence it follows that

$$\sum |a_n| < +\infty, \quad \sum |b_n| < +\infty.$$

The theorem is proved.

§ 62. The Cantor–Lebesgue theorem

We will now consider the coefficients of a trigonometric series, if it converges not absolutely but simply in a set of measure greater than zero.

Here we have

THE CANTOR–LEBESGUE THEOREM. *If a trigonometric series converges in a set E, $mE > 0$, then its coefficients tend to zero.*

In fact, if

$$\sum \varrho_n \cos(nx - \alpha_n) \quad (62.1)$$

converges in E, $mE > 0$, then we have

$$\lim_{n\to\infty} \varrho_n \cos(nx - \alpha_n) = 0 \quad \text{for} \quad x \in E.$$

If a sequence $n_1, n_2, \ldots, n_k, \ldots$ is found, such that

$$\varrho_{n_k} \geqslant \delta > 0, \quad (62.2)$$

then we evidently have
$$\lim_{k\to\infty} \cos(n_k x - \alpha_{n_k}) = 0, \quad x \in E.$$

We will prove that this is not possible. Indeed, then we would have
$$\lim_{k\to\infty} \cos^2(n_k x - \alpha_{n_k}) = 0, \quad x \in E.$$

According to Lebesgue's theorem on the validity of passage to the limit under the integral sign for all-bounded functions, we have, integrating for the set E
$$\lim_{k\to\infty} \int_E \cos^2(n_k x - \alpha_{n_k}) = 0.$$

But since by a similar argument to that in § 61, we have
$$\lim_{k\to\infty} \int_E \cos^2(n_k x - \alpha_{n_k})\, dx = \frac{1}{2} mE,$$

and $mE > 0$, then we arrive at a contradiction.

Consequently, it would be impossible to assume (62.2), therefore
$$\lim_{n\to 0} \varrho_n = 0, \tag{62.3}$$
and the theorem is proved.

Note. The name of this theorem is explained by the fact that Cantor proved it for the case when the series converges in some interval $[a, b]$ and Lebesgue generalized it for the case of any set of positive measure. We think it appropriate here to prove Cantor's theorem separately, as it does not require a knowledge of Lebesgue's integral.

Thus, let the series (62.1) converge in some interval $[a, b]$. For convenience we will rewrite it in the form
$$\sum \varrho_n \cos n(x - \alpha_n). \tag{62.4}$$

It is required to prove that $\varrho_n \to 0$. We will show that this is untrue; then $\delta > 0$ can be found such that
$$\varrho_n \geqslant \delta \tag{62.5}$$
for an infinite set of values of n.

We shall denote the length of the interval $[a, b]$ by d. When x runs through $[a, b]$, then $x - \alpha_n$ runs through an interval of length d. Taking n, such that $n_1 d > 2\pi$, we see that $\cos n_1(x - \alpha_{n_1})$ can run through all its values, while x runs through $[a, b]$, which means that it is possible to find an interval $[a_1, b_1]$ within $[a, b]$ such that this cosine $\geqslant \frac{1}{2}$. If n is chosen so that (62.5) is satisfied, then
$$\varrho_{n_1} \cos n_1(x - \alpha_{n_1}) \geqslant \frac{\delta}{2}, \quad a_1 \leqslant x \leqslant b_1.$$

Let $d_1 = b_1 - a_1$. Arguing in the same way as before, we can choose n_2 so that (62.5) is satisfied for it and so that $n_2 d_1 > 2\pi$, then in the interval $[a_1, b_1]$ an interval $[a_2, b_2]$ is found for which $\cos n_2(x - \alpha_{n_2}) \geqslant \frac{1}{2}$ and therefore
$$\varrho n_2 \cos n_2(x - \alpha_{n_2}) \geqslant \frac{\delta}{2}, \quad a_2 \leqslant x \leqslant b_2.$$

This process can continue indefinitely, since the numbers n satisfying the inequality (62.5) belong to an infinite set. We obtain a sequence of intervals $[a_k, b_k]$, enclosed within one another, whilst

$$\varrho_{n_k} \cos n_k(x - \alpha_{n_k}) \geqslant \frac{\delta}{2}. \tag{62.6}$$

There exists a point ξ, which belongs to all these intervals simultaneously. At this point ξ the inequality (62.6) is fulfilled for all $k (k = 1, 2, \ldots)$ and therefore

$$\lim_{n\to\infty} \varrho_n \cos n(x - \alpha_n) \neq 0,$$

which means that the series $\sum \varrho_n \cos n(x - \alpha_n)$ should diverge at the point ξ. However, ξ lies in the interval $[a, b]$ where the series converges and we arrive at a contradiction.

§ 63. An example of an everywhere divergent series with coefficients tending to zero

The question arises whether a trigonometric series with coefficients tending to zero converges in a set of positive measure. This problem was set by Fatou[1] and the first answer to it was given by Lusin[1], who gave the example of a trigonometric series with coefficients tending to zero and divergence almost everywhere (more detail will be given in §§ 1 and 2, Chapter VII). Then Steinhaus[1] gave the example of a trigonometric series with coefficients tending to zero and divergent at every point.

Here we will describe an example of Steinhaus given in a later report[5].

Consider the series

$$\sum_{k=3}^{\infty} \frac{\cos k(x - \ln \ln k)}{\ln k}. \tag{63.1}$$

Let $l_k = [\ln k]$, $v_k = \ln \ln k$ and

$$g_n(x) = \sum_{k=n+1}^{n+l_n} \frac{\cos k(x - v_k)}{\ln k}; \quad g_n = \sum_{k=n+1}^{n+l_n} \frac{1}{\ln k}.$$

First we note that

$$g_n - g_n(x) = \sum_{k=n+1}^{n+l_n} \frac{1}{\ln k}[1 - \cos k(x - v_k)] = 2 \sum_{k=n+1}^{n+l_n} \frac{\sin^2 k\left(\frac{x - v_k}{2}\right)}{\ln k},$$

whence

$$0 \leqslant g_n - g_n(x) \leqslant \frac{1}{2 \ln n} \sum_{k=n+1}^{n+l_n} k^2 (x - v_k)^2,$$

since $|\sin u| \leqslant |u|$. Let $v_n \leqslant x \leqslant v_{n+1}$ $(n \geqslant 3)$; then for $n + 1 \leqslant k \leqslant n + l_n$ we have, because of the monotonic increase in the numbers v_k:

$$v_n < v_k \leqslant v_{n+l_n},$$

and therefore

$$|x - v_k| \leqslant v_{n+l_n} - v_n.$$

Applying the mean value theorem to the difference $v_{n+l_n} - v_n = \ln\ln(n + l_n) - \ln\ln n$, we find that

$$|x - v_k| \leqslant \frac{l_n}{n\ln n} \leqslant \frac{1}{n},$$

and therefore for $v_n \leqslant x \leqslant v_{n+1}$

$$g_n - g_n(x) \leqslant \frac{1}{2\ln n}\frac{1}{n^2} l_n(n + l_n)^2 \leqslant \frac{1}{2}\left(1 + \frac{l_n}{n}\right)^2. \tag{63.2}$$

The right-hand side of the inequality (63.2) tends to $\frac{1}{2}$ as $n \to \infty$; therefore, for any ε we can find N such that

$$0 \leqslant g_n - g_n(x) \leqslant \frac{1}{2} + \varepsilon \quad \text{for} \quad n \geqslant N. \tag{63.3}$$

On the other hand

$$g_n \geqslant \frac{l_n}{\ln(n + l_n)} \to 1 \quad \text{as} \quad n \to \infty,$$

therefore

$$g_n \geqslant 1 - \varepsilon \quad \text{for} \quad n \geqslant N, \tag{63.4}$$

if N is sufficiently great. If $\varepsilon < \frac{1}{8}$ is taken, then from (63.3) and (63.4)

$$g_n(x) > \frac{1}{2} - 2\varepsilon > \frac{1}{4} \quad \text{for} \quad v_n \leqslant x \leqslant v_{n+1} \quad \text{and} \quad n \geqslant N. \tag{63.5}$$

Now let x be any point of the interval $[0, 2\pi]$. Let us prove that there exists an infinite set of those values of n for which $g_n(x) > \frac{1}{4}$. In fact, if we mark off the points $v_3, v_4, \ldots, v_n$, on the abscissa axis, then they tend monotonically to infinity, which means that the intervals $[v_n, v_{n+1}]$ $(n \geqslant 3)$ cover the whole of the abscissa axis.

Therefore, every point of the type $x + p\,.\,2\pi$ certainly lies within some interval of the type $[v_n, v_{n+1}]$; but $g_n(x + p\,.\,2\pi) = g_n(x)$ and therefore at the point x the inequality (63.5) is satisfied, if $n \geqslant N$.

But for sufficiently large p the inequality $x + p\,.\,2\pi < v_{n+1}$ requires n to be sufficiently large, therefore, for an infinite set of values of $n \geqslant N$ we will indeed have $g_n(x) > \frac{1}{4}$. This means that in the series (63.1) under consideration there is an infinite set of "segments" in which the sum of the terms has a value exceeding $\frac{1}{4}$ and therefore the series diverges. Since this has been proved for any x in $[0, 2\pi]$, then the series diverges at every point.

§ 64. A study of the convergence of one class of trigonometric series

Fatou[1] proved a whole series of important theorems referring to series for which

$$a_n = o\left(\frac{1}{n}\right) \quad \text{and} \quad b_n = o\left(\frac{1}{n}\right). \tag{64.1}$$

But it appears that many of these theorems hold if a weaker requirement is satisfied, namely

$$\tau(n) = \sum_{k=1}^{n} k(|a_k| + |b_k|) = o(n). \tag{64.2}$$

It is clear that (64.2) follows from (64.1) but the converse, generally speaking, does not hold.

Trigonometric series, the coefficients of which satisfy condition (64.2), possess a whole series of interesting properties. They cannot be Fourier series (see Chapter VI, § 3), but this theorem holds:

THEOREM 1. *If the series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

with coefficients satisfying (64.2) *is a Fourier series, then it converges almost everywhere; if it is $\sigma(f)$, where $f(x)$ is continuous, then this series converges uniformly.*

In fact, it is known that if for a trigonometric series $S_n(x)$ are the partial sums and $\sigma_n(x)$ are the Fejér sums, then

$$|S_n(x) - \sigma_n(x)| = \left| \frac{1}{n+1} \sum_{k=1}^{n} k(a_k \cos kx + b_k \sin kx) \right|$$

$$\leqslant \frac{1}{n+1} \sum_{k=1}^{n} k(|a_k| + |b_k|) = o(1) \tag{64.3}$$

because of (64.2). Therefore $S_n(x) - \sigma_n(x) \to 0$ uniformly by virtue of (64.3). But for any Fourier series $\sigma_n(x) \to f(x)$ almost everywhere, therefore $S_n(x) \to f(x)$ almost everywhere. If $f(x)$ is continuous, then $\sigma_n(x) \to f(x)$ uniformly and then $S_n(x) \to f(x)$ uniformly, and the theorem is proved.

As a corollary we deduce the theorem:

THEOREM 2. (Fatou). *If a trigonometric series has coefficients of the form*

$$a_n = o\left(\frac{1}{n}\right) \quad \text{and} \quad b_n = o\left(\frac{1}{n}\right),$$

then it converges almost everywhere.

If moreover it is a Fourier series of a continuous function, then it converges uniformly.

Indeed, it is clear above all that our series is a Fourier series, since $\sum(a_n^2 + b_n^2) < +\infty$. Moreover, as we have already said, (64.3) follows from (64.1), which means that we have conditions of applicability of the preceding theorem.

Note. The hypothesis relating to continuity is an additional requirement and does not follow from (64.1). It is possible to show that functions exist for which the Fourier coefficients satisfy condition (64.1) but, however, they are unbounded in any interval δ, lying in $[-\pi, \pi]$ (see Chapter VIII, § 13).

§ 65. Lacunary sequences and lacunary series

Let us derive some corollaries from Theorem 1, § 64. For this we recall that in Introductory Material, § 4 we defined a sequence of natural numbers $\{n_k\}$ as satisfying condition (L), if

$$\sum_{k=1}^{\infty} \frac{1}{n_k} < +\infty$$

and

$$\sum_{k=m}^{\infty} \frac{1}{n_k} = O\left(\frac{1}{n_m}\right) \quad (m = 1, 2, \ldots).$$

The sequence $\{n_k\}$ is named lacunary if there exists $\lambda > 1$ such that

$$\frac{n_{k+1}}{n_k} > \lambda > 1 \quad (k = 1, 2, \ldots). \tag{65.1}$$

Finally, it was proved that any lacunary sequence satisfies condition (L).

Now we will define lacunary series.

DEFINITION. The series

$$\sum (a_k \cos n_k x + b_k \sin n_k x) \tag{65.2}$$

is named *lacunary* if the natural numbers $\{n_k\}$ form a lacunary sequence (i.e. satisfy the condition (65.1)).

If the sequence $\{n_k\}$ satisfies condition (L) then we will say that series (65.2) is an (L)-series (thus, any lacunary series is an (L)-series, but the converse is generally not the case).

We will prove that if the coefficients of an (L)-series tend to zero, then it belongs to the class of series studied in § 64. Indeed, the function $\tau(n)$ defined in § 64 (see (64.2)) in the given case takes the form

$$\tau(n) = \sum_{k_n \leqslant n} n_k(|a_k| + |b_k|).$$

We will prove that $\tau(n) = o(n)$, then we will have the conditions of § 64. Since $a_k \to 0$ and $b_k \to 0$, then for any $\varepsilon > 0$, p is found such that $|a_k| \leqslant \varepsilon$ and $|b_k| \leqslant \varepsilon$ at $k \geqslant p$. If n_m is the greatest number of the sequence $\{n_k\}$ not exceeding n, then

$$\tau(n) = \sum_{k=1}^{m} n_k(|a_k| + |b_k|) \leqslant \sum_{k=1}^{p} n_k(|a_k| + |b_k|) + 2\varepsilon \sum_{k=p+1}^{m} n_k. \tag{65.3}$$

Since the first term on the right-hand side of (65.3) does not vary with n, then it is possible to make n_0 so large that this term will be less than εn for $n \geqslant n_0$. Then

$$\frac{\tau(n)}{n} \leqslant \varepsilon + \frac{2\varepsilon}{n_m} \sum_{k=p+1}^{m} n_k < c\varepsilon \quad (\text{for } \; n \geqslant n_0),$$

where c is a constant because for the sequences $\{n_k\}$ satisfying condition (L) we have

$$\sum_{k=1}^{m} n_k = O(n_m)$$

(see Introductory Material, § 4).

Thus

$$\tau(n) \leqslant c\,\varepsilon_n,$$

and since ε is as small as desired, then

$$\tau(n) = o(n)$$

and our statement is proved.

From the proved statement and Theorem 1, § 64, we immediately obtain:

COROLLARY 1. *If an (L)-series is a Fourier series, then it converges almost everywhere; if it is a Fourier series for a continuous function, it converges uniformly.*

From the note made above that any lacunary sequence satisfies condition (L) and from Corollary 1 we obtain Kolmogorov's theorem[6].

If a lacunary series is a Fourier series, then it converges almost everywhere.

Moreover, from Corollary 1 we also immediately obtain: *if a lacunary series is a Fourier series for a continuous function, then it converges uniformly.*

In Chapter XI, § 6 a stronger assertion will be proved, namely, that under the given conditions, the series should also converge absolutely.

We will now prove yet another theorem relating to sequences satisfying condition (L).

THEOREM. *Let $\{n_k\}$ be a sequence satisfying condition (L) and $f(x)$ be a function with an integrable square. Then*

$$S_{n_k}(x) \to f(x) \text{ almost everywhere as } k \to \infty.$$

Proof. Let

$$\frac{a_0}{2} + \sum (a_n \cos nx + b_n \sin nx)$$

be the Fourier series of $f(x)$, $S_n(x)$ and $\sigma_n(x)$ be its partial and Fejér sums. Since $\sigma_n(x) \to f(x)$ as $n \to \infty$ almost everywhere, then it is sufficient to prove that $S_{n_k}(x) - \sigma_{n_k}(x) \to 0$ almost everywhere.

We will prove that

$$\sum_{k=1}^{\infty} \int_{-\pi}^{\pi} [\sigma_{n_k}(x) - S_{n_k}(x)]^2 dx < +\infty, \tag{65.3'}$$

then according to Lebesgue's theorem (see Introductory Material, § 14) the series

$$\sum_{k=1}^{\infty} [\sigma_{n_k}(x) - S_{n_k}(x)]^2,$$

will converge almost everywhere and therefore its general term will tend to zero.

Thus, it remains to prove the convergence of series (65.3'). As is known,

$$S_n(x) - \sigma_n(x) = \frac{1}{n+1} \sum_{m=0}^{n} m(a_m \cos mx + b_m \sin mx).$$

Therefore due to Parseval's equality

$$\frac{1}{\pi} \int_{-\pi}^{\pi} [\sigma_{n_k}(x) - S_{n_k}(x)]^2 dx = \frac{1}{(n_k+1)^2} \sum_{m=0}^{n_k} m^2 (a_m^2 + b_m^2).$$

Let us estimate the sum of the first p terms of series (65.3'); we have

$$\frac{1}{\pi} \sum_{k=1}^{p} \int_{-\pi}^{\pi} [S_{n_k}(x) - \sigma_{n_k}(x)]^2 dx = \sum_{k=1}^{p} \frac{1}{(n_k+1)^2} \sum_{m=0}^{n_k} m^2 (a_m^2 + b_m^2)$$

$$< \sum_{k=1}^{p} \frac{1}{n_k^2} \sum_{m=0}^{n_k} m^2 (a_m^2 + b_m^2). \tag{65.4}$$

To shorten the working-out, we introduce the notation

$$v_m = m^2(a_m^2 + b_m^2).$$

We have

$$\sum_{k=1}^{p} \frac{1}{n_k^2} \sum_{m=0}^{n_k} v_m = \sum_{k=1}^{n_1} v_k \sum_{m=1}^{p} \frac{1}{n_m^2} + \sum_{n_1+1}^{n_2} v_k \sum_{m=2}^{p} \frac{1}{n_m^2} + \cdots + \sum_{k=n_{p-1}+1}^{n_p} v_k \frac{1}{n_p^2}. \tag{65.5}$$

But the sequence $\{n_k\}$ satisfies condition (L), and therefore $\{n_k^2\}$ does so, too (see Introductory Material, § 4) and therefore

$$\sum_{k=m}^{\infty} \frac{1}{n_k^2} < C\frac{1}{n_m^2}, \tag{65.6}$$

where C is a constant. But then, supposing $n_0 = 0$, we find from (65.5) and (65.6) that

$$\sum_{k=1}^{p} \frac{1}{n_k^2} \sum_{m=0}^{n_k} v_m < C \sum_{k=1}^{p} (v_{n_{k-1}+1} + \cdots + v_{n_k}) \frac{1}{n_k^2}.$$

Finally, it is clear from the definition of v_m that

$$v_{n_{k-1}+1} + \cdots + v_{n_k} < n_k^2 \sum_{n_{k-1}+1}^{n_k} (a_m^2 + b_m^2) \tag{65.7}$$

$$\sum_{k=1}^{p} \frac{1}{n_k^2} \sum_{m=0}^{n_k} v_m < c \sum_{k=1}^{p} \sum_{n_{k-1}+1}^{n_k} (a_m^2 + b_m^2) = c \sum_{m=1}^{n_p} (a_m^2 + b_m^2) \leqslant c \int_{-\pi}^{\pi} f^2(x)dx$$

for any p. Hence the convergence of the series on the right-hand side of (65.4) follows and this concludes the proof of the theorem.

COROLLARY 2. From the statement just, proved, Kolmogorov's[6] theorem follows. *If $\{n_k\}$ is a lacunary sequence and $f(x) \in L^2$, then*

$$S_{n_k}(x) \to f(x) \text{ almost everywhere.}$$

66. Smooth functions

For the further investigation of the series which we considered in § 64 and also in many other problems, it will be useful to understand the concept of a smooth function.

DEFINITION. The function $F(x)$ is said to be *smooth at the point* x, if

$$\frac{F(x+h) + F(x-h) - 2F(x)}{h} \to 0 \quad \text{as} \quad h \to 0. \tag{66.1}$$

Defining for brevity's sake

$$\Delta_h^2 F = F(x+h) + F(x-h) - 2F(x),$$

we can say that the smoothness of $F(x)$ is characterized by the equality

$$\Delta_h^2 F = o(h).$$

If equality (66.1) is fulfilled uniformly relative to x for some interval $[a, b]$, then we say that $F(x)$ is *uniformly smooth* in this interval.

The word "smooth" is evidently introduced to represent the following idea: if $F(x)$ is smooth at some point, then this point cannot be angular. Indeed, if $\Delta_h^2 F = o(h)$ at the point x, then

$$\frac{F(x+h) + F(x-h) - 2F(x)}{h} = \frac{F(x+h) - F(x)}{h} - \frac{F(x-h) - F(x)}{-h}$$
$$= o(1),$$

i.e. if a derivative exists on the right of the point x, then a derivative also exists on the left and they should be equal to one another. Moreover, if $F(x)$ is smooth at some point, then at the same point

$$D^+F = D^-F = \overline{D}F \quad \text{and} \quad D_+F = D_-F = \underline{D}F,$$

where D^+F and D^-F denote the upper right and upper left derivatives and $\overline{D}F$ the upper derivative (when they are equal); similar notation is used for the lower derivatives.

We note that if $f(x)$ is continuous at the point x or only "symmetrically continuous," i.e.

$$f(x_0 + h) - f(x_0 - h) \to 0 \quad \text{as} \quad h \to 0,$$

then the primitive $F(x)$ of $f(x)$ satisfies the condition of smoothness at this point, since

$$F(x_0 + h) + F(x_0 - h) - 2F(x_0) = \int_0^h [f(x_0 + t) - f(x_0 - t)]\, dt = o(h)$$

as $h \to 0$.

However, smooth functions, in spite of their name, should not necessarily possess a derivative almost everywhere; moreover, they can be devoid of a derivative almost everywhere, as we will see later (see Chapter XI, § 4). However, the following theorem holds:

THEOREM 1. *If $F(x)$ is continuous and smooth in some interval (a, b), then it possesses a derivative $F'(x)$ in a set E of the power of the continuum in any interval (α, β) lying within (a, b).*

To prove this we first note that if the function $F(x)$ possesses a maximum or minimum at some point x_0 within the interval $[a, b]$, then $F'(x_0)$ exists and equals zero. In fact, we have

$$\frac{F(x_0 + h) + F(x_0 - h) - 2F(x_0)}{h} = \frac{F(x_0 + h) - F(x_0)}{h} + \frac{F(x_0 - h) - F(x_0)}{h}. \tag{66.2}$$

But at the maximum (or minimum) point neither terms on the right-hand side of (66.2) are positive for sufficiently small $h > 0$ (or correspondingly negative). Therefore since their sum tends to zero, it follows that each of them tends to zero, and then $F'(x_0)$ exists and equals zero.

Now let $[\alpha, \beta]$ be any interval within (a, b) and $L(x) = mx + n$ be a linear function coinciding with $F(x)$ at $x = \alpha$ and $x = \beta$. The difference $g(x) = F(x) - L(x)$ is a smooth function returning to zero at the end points α and β. This means that $g(x)$

has an absolute maximum or minimum at some point x_0 inside (α, β). Therefore, $g'(x_0) = 0$, which means that $F'(x_0)$ exists and equals m. Hence, in particular, it follows that for continuous and smooth functions the first mean value theorem holds, i.e.

$$F(b) - F(a) = (b - a)F'(\xi), \quad a < \xi < b.$$

We have proved that in any $[\alpha, \beta]$ within (a, b) there are points where $F'(x)$ exists. But it can be proved moreover that the set of these points is of the power of the continuum. Indeed, let γ be given such that $\alpha < \gamma < \beta$. A point x_0 is found in (α, γ) where $F'(x)$ exists and equals the tangent of the angle of inclination of the chord connecting the points $(\alpha, F(\alpha))$ and $(\gamma, F(\gamma))$. If the inclinations corresponding to different γ are different, then the corresponding points x_0 are also different. But if the curve $y = F(x)$ is not a rectilinear interval in (α, β) (if this were the case the theorem has already been proved), then the magnitudes of the tangents of these slopes form the interval, i.e. their set is of the power of the continuum and therefore the points of differentiability of $F(x)$ belong to a set with power of continuum in the whole interval. The theorem is proved.

We will now give a definition for later use.

Definition. We say that *the function $f(x)$ possesses the property D in some set E*, if for any two $\alpha \in E$ and $\beta \in E$ and for any number C, contained between $f(\alpha)$ and $f(\beta)$, a point $\gamma \in E$ is found lying between α and β such that $f(\gamma) = C$.

The letter D is derived from Darboux's name, since he noticed that this property was possessed not only by functions continuous in some interval but also by some discontinuous functions; in particular, if $f(x)$ is an exact derivative, i.e. if $F(x)$ exists such that $f(x) = F'(x)$ at every point of some interval, then it possesses property D in that interval.

Let us prove a theorem.

Theorem 2. *If $F(x)$ is continuous and smooth in some interval (a, b), then its derivative $F'(x)$ possesses property D in the set E of all the points where it exists.*

This set E, as we can see from Theorem 1, is not only not empty but is of the power of the continuum in every interval $[\alpha, \beta]$ within (a, b).

Let $\alpha \in E$, $\beta \in E$

$$A = F'(\alpha), \quad B = F'(\beta)$$

and let C be contained between A and B; for example, let us define $A < C < B$. We should prove the existence of x_0, $\alpha < x_0 < \beta$, $x_0 \in E$, such that $F'(x_0) = C$. If we subtract Cx from $F(x)$ then it is possible to assume $C = 0$, and then $A < 0 < B$.

Let us suppose for a fixed h that

$$g(x) = \frac{F(x + h) - F(x)}{h}.$$

We choose h such that $0 < h < b - \beta$ and moreover we suppose its range to be so small that

$$g(\alpha) < 0, \quad g(\beta) > 0, \quad \frac{F(\beta) - F(\beta - h)}{h} > 0.$$

Since $g(x)$ is continuous in $[\alpha, \beta]$, then in this interval $[\alpha, \beta]$ there are points where it becomes zero. Let γ be the furthest left of these points. From

$$g(\gamma) = \frac{F(\gamma + h) - F(\gamma)}{h} = 0$$

it follows that $F(\gamma + h) = F(\gamma)$. If x_0 is a point in $(\gamma, \gamma + h)$, where $F(x)$ reaches a maximum or minimum, then $F'(x_0) = 0 = C$. But since

$$g(\alpha) < 0 \quad \text{and} \quad g(\beta - h) = \frac{F(\beta) - F(\beta - h)}{h} > 0$$

because of the given choice of h, then

$$\alpha < \gamma < \beta - h$$

which means that $(\gamma, \gamma + h)$ lies inside $[\alpha, \beta]$, therefore x_0 is also inside $[\alpha, \beta]$. Moreover, since $F'(x_0)$ exists, then $x_0 \in E$. Thus we have found a point $x_0 \in E$, where $F'(x_0) = C$, and the proof is concluded.

We will apply the results obtained to the investigation of the behaviour of the sum of the trigonometric series considered in § 64. First we will prove this theorem:

THEOREM 3. *If the coefficients of the series*

$$\frac{a_0}{2} + \sum (a_n \cos nx + b_n \sin nx) \tag{66.3}$$

satisfy the condition

$$\tau(n) = \sum_{k=1}^{n} k(|a_k| + |b_k|) = o(n), \tag{66.4}$$

then the sum of the integrated series

$$F(x) = \frac{a_0}{2} x + C - \sum_{n=1}^{\infty} \frac{b_n \cos nx - a_n \sin nx}{n} \tag{66.5}$$

is a function which is continuous and uniformly smooth in $[0, 2\pi]$. *The series* (66.3) *converges at those points and only at those points where* $F'(x)$ *exists and besides, if* $N = [1/h]$ *then the equality*

$$\frac{F(x + h) - F(x - h)}{2h} - \left[\frac{a_0}{2} + \sum_{k=1}^{N} (a_k \cos kx + b_k \sin kx)\right] \to 0 \quad as \quad h \to 0 \tag{66.5}$$

occurs uniformly relative to x in $[0, 2\pi]$.

In order to be able to speak correctly of the sum of an integrated series, it must be proved that it converges. But because

$$|a_k| + |b_k| = \frac{\tau(k) - \tau(k - 1)}{k} \quad (k = 1, 2, \ldots),$$

then the series (66.5) is majorized by the series

$$\sum_{k=1}^{\infty} \frac{|a_k| + |b_k|}{k} = \sum_{k=1}^{\infty} \frac{\tau(k) - \tau(k-1)}{k^2} \leqslant \sum_{k=1}^{\infty} \tau(k) \left(\frac{1}{k^2} - \frac{1}{(k+1)^2}\right)$$
$$= O\left(\sum_{k=1}^{\infty} \frac{1}{k^2}\right) < +\infty$$

(we applied Abel's transformation here). Hence it is clear that series (66.5) converges absolutely and uniformly. Let $F(x)$ be its sum, therefore, it is continuous in $[0, 2\pi]$.

Now in order to prove the theorem we suppose

$$A_k = a_k \cos kx + b_k \sin kx, \quad B_k = b_k \cos kx - a_k \sin kx.$$

Then

$$\frac{F(x+h) - F(x-h)}{2h} - S_N(x) = \sum_{k=1}^{N} A_k \left(\frac{\sin kh}{kh} - 1\right) + \sum_{k=N+1}^{\infty} A_k \frac{\sin kh}{kh}$$
$$= P + Q.$$

Since in the neighbourhood of the point $u = 0$, we have

$$\left|\frac{\sin u}{u} - 1\right| = O(u^2) < C|u|,$$

then

$$|P| < C|h| \sum_{k=1}^{N} (|a_k| + |b_k|)\, k \leqslant C \frac{1}{N} \tau(N) = o(1),$$

$$|Q| \leqslant \frac{1}{|h|} \sum_{k=N+1}^{\infty} \frac{|a_k| + |b_k|}{k} \leqslant \frac{1}{|h|} \sum_{k=N+1}^{\infty} \tau_k \left(\frac{1}{k^2} - \frac{1}{(k+1)^2}\right)$$
$$= \frac{1}{|h|} \sum_{k=N+1}^{\infty} o\left(\frac{1}{k^2}\right) = \frac{1}{|h|} o\left(\frac{1}{N}\right) = o(1)$$

and thus (66.2) is actually fulfilled, and moreover uniformly relative to x in $[0, 2\pi]$.

Similarly we have

$$\frac{F(x+2h) + F(x-2h) - 2F(x)}{4h} = \sum_{k=1}^{\infty} B_k \frac{\sin^2 kh}{kh}$$
$$= \sum_{k=1}^{N} B_k \frac{\sin^2 kh}{kh} + \sum_{k=N+1}^{\infty} B_k \frac{\sin^2 kh}{kh}$$
$$= P_1 + Q_1.$$

Since $|\sin u| \leqslant |u|$, then

$$|P_1| \leqslant |h| \sum_{k=1}^{N} |B_k|\, k \leqslant |h| \sum_{k=1}^{N} (|a_k| + |b_k|)\, k = |h|\, \tau_N = o(1),$$

$$|Q_1| \leqslant \frac{1}{|h|} \sum_{k=N+1}^{\infty} \frac{|a_k| + |b_k|}{k} = o(1),$$

as we have already seen in estimating Q; therefore

$$F(x + 2h) + F(x - 2h) - 2F(x) = o(h),$$

i.e. $F(x)$ is uniformly smooth in $[0, 2\pi]$.

Finally, from (66.6) it is clear that series (66.3) converges at those points and only at those points where a symmetrical derivative of $F(x)$ exists, i.e.

$$\lim_{h\to 0} \frac{F(x + h) - F(x - h)}{2h}$$

and the sum $S(x)$ of this series equals its symmetrical derivative.

But since for smooth functions where a symmetrical derivative exists, the normal derivative also exists, then the latter part of the theorem is proved.

Note. The proved theorem is valid for lacunary series with coefficients tending to zero, since for them the conditions of Theorem 3 (see § 65) are fulfilled.

COROLLARY 1. *If for a trigonometric series the conditions of Theorem 3 are fulfilled, then the series converges in a set of the power of the continuum in any interval* (a, b) $\in [0, 2\pi]$ *and its sum* $S(x)$ *possesses property D in a set of those points where it exists.*

In particular, this property is possessed by any lacunary series, provided its coefficients tend to zero.

Indeed, by virtue of Theorem 3 $S(x)$ exists where and only where $F'(x)$ exists for a smooth function $F(x)$ defined by equality (66.5) and moreover $S(x) = F'(x)$; then, reference must be made to Theorem 2 and the proof is concluded.

COROLLARY 2. *If the coefficients of a trigonometric series satisfy the conditions of Theorem 3, then its sum cannot have points of discontinuity of the first kind.*

Indeed, in the neighbourhood of a point of discontinuity property D would not be fulfilled.

For the case when the coefficients satisfy a stronger requirement

$$a_n = o\left(\frac{1}{n}\right), \quad b_n = o\left(\frac{1}{n}\right), \tag{66.7}$$

we have already obtained a similar result (see § 42).

Note 1. From Theorem 3 it is possible to obtain a new proof of Fatou's theorem (see § 64) that *a series with coefficients satisfying* (66.7) *converges almost everywhere.* Indeed, since from (66.7) it follows that $\sum (a_n^2 + b_n^2) < +\infty$, then the series $F(x)$ is a Fourier series. Therefore the sum $F(x)$ of an integrated series is an absolutely continuous function (see § 40) and therefore, $F'(x)$ exists almost everywhere. Then due to Theorem 3, the series (66.3) converges almost everywhere.

Note 2. If condition (66.4) only is fulfilled and not condition (66.7), then series (66.3) can even diverge almost everywhere. We meet examples of this kind in § 3 of Chapter XI.

§ 67. The Schwarz second derivative

The concept of a smooth function studied by us in § 66 will play a great part in later work; but before turning to its application, we must first introduce yet another new concept.

DEFINITION. Let the function $F(x)$ be defined in some neighbourhood of the point x; if the limit of the expression

$$\frac{F(x+h)+F(x-h)-2F(x)}{h^2} \tag{67.1}$$

exists as $h \to 0$, then it is said that $F(x)$ possesses at the point x *a Schwarz second derivative* and we write

$$D^2F(x) = \lim_{h\to 0} \frac{F(x+h)+F(x-h)-2F(x)}{h^2}. \tag{67.2}$$

If the relation (67.1) does not tend to a limit as $h \to 0$, then the values

$$\bar{D}^2F(x) = \overline{\lim_{h\to 0}} \frac{F(x+h)+F(x-h)-2F(x)}{h^2}$$

and

$$\underline{D}^2F(x) = \underline{\lim}_{h\to 0} \frac{F(x+h)+F(x-h)-2F(x)}{h^2}$$

are called respectively *the upper and lower Schwarz derivatives at the point x.*

We will show that if $F(x)$ possesses a normal second derivative $F''(x)$ at the point x, then $D^2F(x)$ exists and

$$D^2F(x) = F''(x). \tag{67.3}$$

Indeed, if $F''(x)$ exists at the point x, then $F'(x)$ is continuous at the point x and therefore $F'(x)$ is bounded in the neighbourhood of the point x. It is clear that

$$\Delta_h^2F = F(x+h)+F(x-h)-2F(x) = \int_0^h [F'(x+t)-F'(x-t)]dt. \tag{67.4}$$

Hence

$$\left|\frac{\Delta_h^2F}{h^2} - F''(x)\right| = \left|\int_0^h \frac{2t}{h^2}\left[\frac{F'(x+t)-F'(x-t)}{2t} - F''(x)\right]dt\right|$$

$$\leqslant \max_{t\in(0,h)}\left|\frac{F'(x+t)-F'(x-t)}{2t} - F''(x)\right| \to 0 \quad \text{as} \quad h \to 0,$$

i.e. (67.3) is proved.

On the other hand it is clear that $D^2F(x)$ can exist without $F''(x)$ existing; for example, if $F(x)$ is a continuous odd function, then at the point $x = 0$ we have

$$F(x+h)+F(x-h)-2F(x) = F(h)+F(-h) = 0$$

for all h, which means that $D^2F = 0$ at $x = 0$, whilst $F''(0)$ cannot exist, if we only require that $F(x)$ be continuous and odd.

Thus, the Schwarz second derivative is a direct generalization of the normal second derivative.

We now note that, as in the case of the normal second derivative, we have: if x is a maximum point and $D^2F(x)$ exists at it, then $D^2F(x) \leqslant 0$ and at a minimum

$D^2 F(x) \geqslant 0$. This follows from the fact that $\Delta_h^2 F(x) \leqslant 0$ for sufficiently small h at the maximum point and $\Delta_h^2 F(x) \geqslant 0$ at the minimum point.

The analogy continues still further. Thus the following theorem holds.

THEOREM. *If $F(x)$ is continuous in $[a, b]$ and $D^2 F(x) \equiv 0$ in $a < x < b$, then $F(x)$ is linear in this interval.*

In order to prove this, take any $\varepsilon > 0$ and consider an auxiliary function

$$\varphi(x) = F(x) - F(a) - \frac{F(b) - F(a)}{b - a}(x - a) + \varepsilon(x - a)(x - b).$$

It is clear that $\varphi(a) = \varphi(b) = 0$. We will prove that it cannot assume positive values in $[a, b]$. Indeed, if this were the case, then because of the continuity of $\varphi(x)$ it would attain its maximum somewhere within $[a, b]$, i.e. a point x_0 would be found in this interval, where it would be known that $D^2\varphi(x_0) \leqslant 0$. But, on the other hand,

$$D^2\varphi(x_0) = D^2 F(x_0) + 2\varepsilon,$$

since the Schwarz second derivative of the sum equals the sum of the Schwarz second derivatives, and the term $\varepsilon(x - a)(x - b)$ has the normal second derivative equal to 2ε, which means that the Schwarz second derivative has exactly the same magnitude.

But $D^2\varphi(x_0) \leqslant 0$, $D^2 F(x_0) = 0$, and we obtain $\varepsilon \leqslant 0$ which contradicts the choice of ε.

Thus $\varphi(x) \leqslant 0$ everywhere in $[a, b]$, i.e.

$$F(x) - F(a) - \frac{F(b) - F(a)}{b - a}(x - a) \leqslant \varepsilon(x - a)(b - x) \leqslant \varepsilon(b - a)^2.$$

If we were to put a minus sign in front of ε in the expression for $\varphi(x)$, we would prove in exactly the same way that $\varphi(x) \geqslant 0$ everywhere, i.e.

$$F(x) - F(a) - \frac{F(b) - F(a)}{b - a}(x - a) \geqslant -\varepsilon(x - a)(b - x) \geqslant -\varepsilon(b - a)^2.$$

Therefore

$$\left| F(x) - F(a) - \frac{F(b) - F(a)}{b - a}(x - a) \right| \leqslant \varepsilon(b - a)^2. \tag{67.5}$$

But ε is quite arbitrary, therefore the left-hand side of the inequality (67.5) should be equal to zero, whence

$$F(x) = F(a) + \frac{F(b) - F(a)}{b - a}(x - a),$$

which means that $F(x)$ is linear. The theorem has been proved.

We will now apply the concept of the Schwarz second derivative to a method of summation of trigonometric series.

§ 68. Riemann's method of summation

Let us consider the trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \tag{68.1}$$

the coefficients of which tend to zero (or are only bounded). Then, integrating it twice term-by-term, we obtain

$$\frac{a_0}{4} x^2 + Cx + D - \sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^2}.$$

It is clear that this series converges absolutely and uniformly (because a_n and b_n are bounded); let us denote its sum by $F(x)$. It is a continuous function which we will name the *Riemann function* for the trigonometric series (68.1). Thus

$$F(x) = \frac{a_0}{4} x^2 + Cx + D - \sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^2}. \tag{68.2}$$

We assume that at some point x_0 the function $F(x)$ possesses a Schwarz derivative $D^2 F(x_0)$. Then we can say that *the series* (68.1) *is summable at the point* x_0 *by Riemann's method* and its Riemann sum equals $D^2 F(x_0)$.

In order to verify this statement, we will prove Riemann's theorem:

THEOREM 1. *If a trigonometric series with coefficients tending to zero converges at a point x_0 to a value S, then it is summable at this point by Riemann's method to the same value S.*

To prove this, we note first of all that it immediately follows from formula (68.2) after elementary trigonometric transformations that

$$\frac{F(x_0 + 2h) + F(x_0 - 2h) - 2F(x_0)}{4h^2}$$
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx_0 + b_n \sin nx_0) \left(\frac{\sin nh}{nh}\right)^2. \tag{68.3}$$

For brevity's sake we assume

$$A_0 = \frac{a_0}{2}, \quad A_n = a_n \cos nx_0 + b_n \sin nx_0.$$

From formula (68.3) it is immediately evident that for the summability of series (68.1) by Riemann's method at the point x_0 to a value S it is necessary and sufficient that

$$\lim_{h \to 0} \left[A_0 + \sum_{n=1}^{\infty} A_n \left(\frac{\sin nh}{nh}\right)^2\right] = S.$$

Thus, Theorem 1 will only be proved when we prove Theorem 2:

THEOREM 2. *Let the series* $A_0 + \sum_{n=1}^{\infty} A_n$ *converge and S be its sum; then*

$$\lim_{h\to 0}\left[A_0 + \sum_{n=1}^{\infty} A_n \left(\frac{\sin nh}{nh}\right)^2\right] = S. \tag{68.4}$$

We will now prove this latter assertion. Let us suppose

$$R_n = \sum_{k=n+1}^{\infty} A_k.$$

From the convergence of the series $\sum A_n$ it follows that for any $\varepsilon > 0$ it is possible to find N such that

$$|R_n| < \varepsilon \quad \text{at} \quad n \geqslant N. \tag{68.5}$$

We now write

$$A_0 + \sum_{n=1}^{\infty} A_n \left(\frac{\sin nh}{nh}\right)^2 = A_0 + \sum_{n=1}^{N} A_n \left(\frac{\sin nh}{nh}\right)^2 + \sum_{N+1}^{\infty} A_n \left(\frac{\sin nh}{nh}\right)^2. \tag{68.6}$$

If n is fixed and $h \to 0$, then $\dfrac{\sin nh}{nh} \to 1$, and therefore for sufficiently small h

$$\left| A_0 + \sum_{n=1}^{N} A_n \left(\frac{\sin nh}{nh}\right)^2 - (A_0 + A_1 + \cdots + A_N)\right| < \varepsilon. \tag{68.7}$$

Moreover,

$$\left| S - \sum_{k=0}^{N} A_k \right| = |R_N| < \varepsilon \tag{68.8}$$

due to (68.5), and therefore from (68.7) and (68.8)

$$\left| A_0 + \sum_{n=1}^{\infty} A_n \left(\frac{\sin nh}{nh}\right)^2 - S \right| < 2\varepsilon, \tag{68.9}$$

if only h becomes sufficiently small.

Thus in order to prove (68.4) it is sufficient to prove that the last term on the right-hand side of formula (68.6) can be made as small as desired as $h \to 0$. But we have $A_n = R_{n-1} - R_n$, which means that

$$\sum_{N+1}^{\infty} A_n \left(\frac{\sin nh}{nh}\right)^2 = \sum_{N+1}^{\infty} (R_{n-1} - R_n) \left(\frac{\sin nh}{nh}\right)^2$$

$$= R_N \left(\frac{\sin (N+1)h}{(N+1)\,h}\right)^2 - \sum_{N+1}^{\infty} R_n \left[\left(\frac{\sin nh}{nh}\right)^2 - \left(\frac{\sin (n+1)h}{(n+1)h}\right)^2\right] \tag{68.10}$$

(Abel's transformation used here is valid, since as $n \to \infty$ and h being any value

$R_n \left(\frac{\sin nh}{nh}\right)^2 \to 0$). But by virtue of (68.5) we obtain from (68.10)

$$\left|\sum_{N+1}^{\infty} A_n \left(\frac{\sin nh}{nh}\right)^2\right| \leqslant \varepsilon + \varepsilon \sum_{N+1}^{\infty} \left|\left(\frac{\sin nh}{nh}\right)^2 - \left(\frac{\sin(n+1)h}{(n+1)h}\right)^2\right|$$

$$= \varepsilon + \varepsilon \sum_{N+1}^{\infty} \left|\int_{nh}^{(n+1)h} \frac{d}{dt}\left(\frac{\sin t}{t}\right)^2 dt\right|$$

$$\leqslant \varepsilon + \varepsilon \int_{(N+1)h}^{\infty} \left|\frac{d}{dt}\left(\frac{\sin t}{t}\right)^2\right| dt < \varepsilon + \varepsilon \int_{0}^{\infty} \left|\frac{d}{dt}\left(\frac{\sin t}{t}\right)^2\right| dt \tag{68.11}$$

and it remains to prove that the last integral is finite, then the whole of the right-hand side of (68.11) is less than $C\varepsilon$, where C is a constant, and since this is true for any h, then it is also true as $h \to 0$. Since

$$\frac{d}{dt}\left(\frac{\sin t}{t}\right)^2 = 2\,\frac{\sin t}{t}\,\frac{t\cos t - \sin t}{t^2},$$

then in the neighbourhood of $t = 0$ the function under the integral sign is bounded, moreover, as $t \to \infty$ we have

$$\left|\frac{d}{dt}\left(\frac{\sin t}{t}\right)^2\right| < 2\,\frac{t+1}{t^3} = O\left(\frac{1}{t^2}\right),$$

and therefore the integral in formula (68.11) does indeed have meaning and the proof is concluded.

Note. In the proof of Theorem 2, we considered the series $\sum_{n=0}^{\infty} A_n$ to be a numerical series, without being concerned with the fact that it was obtained from a given trigonometric series. It can be said in general that *the numerical series $\sum_{n=0}^{\infty} u_n$ is summable by Riemann's method to the value S, if*

$$\lim_{h\to 0}\left[u_0 + \sum_{n=1}^{\infty} u_n \left(\frac{\sin nh}{nh}\right)^2\right] = S.$$

In this case Theorem 2 is a statement that *Riemann's method is regular.*

Now it must be said that *the functional series $\sum u_n(x)$ is summable by Riemann's method uniformly to $S(x)$ in the set E, if*

$$\lim_{n\to\infty}\left[u_0(x) + \sum_{n=1}^{\infty} u_n(x) \left(\frac{\sin nh}{nh}\right)^2\right] = S(x)$$

uniformly relative to x in E.

From the proof of Theorem 2 it is immediately evident that the uniform convergence of $\sum u_n(x)$ in E to $S(x)$ implies its uniform summability by Riemann's method to $S(x)$ in E.

This note will be used essentially in § 71.

We will now return to the study of the Riemann function $F(x)$ and will prove yet another theorem due to Riemann.

THEOREM 3. *If the coefficients of a trigonometric series tend to zero, then its Riemann function is uniformly smooth in* $[-\pi, \pi]$.

This theorem follows quite quickly from the results of § 66. Indeed, if we integrate the series

$$\frac{a_0}{2} + \sum (a_n \cos nx + b_n \sin nx), \tag{68.12}$$

where $a_n \to 0$, $b_n \to 0$, then we obtain a series with coefficients of order $o(1/n)$

$$\frac{a_0}{2} x + C - \sum \frac{b_n \cos nx - a_n \sin nx}{n}. \tag{68.13}$$

Integrating the series (68.13), we obtain according to the theorem of § 66 a series, the sum of which should be uniformly smooth. But this sum $F(x)$ is the sum of a series obtained by the double successive integration of (68.12), and therefore it is also the Riemann function for the series (68.12) and the theorem is proved.

We will use this theorem in § 70 but first we will consider the application of Riemann's method to Fourier series.

§ 69. Application of Riemann's method of summation to Fourier series

Riemann's method, as well as the methods of Fejér and Abel–Poisson, when applied to Fourier series, gives the following result:

THEOREM. *The Fourier series for any summable function $f(x)$ is summable by Riemann's method almost everywhere to this function.*

Indeed, let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \tag{69.1}$$

We have $a_n \to 0$ and $b_n \to 0$, since these are Fourier coefficients. According to the theorem of § 40, the Fourier series can be integrated term by term; in other words, if

$$F(x) = \int_{-\pi}^{x} f(t)\, dt,$$

then

$$F(x) = C + \frac{a_0}{2} x - \sum_{n=1}^{\infty} \frac{b_n \cos nx - a_n \sin nx}{n}, \tag{69.2}$$

whilst because of the absolute continuity of $F(x)$ the series (69.2) converges everywhere to it and even uniformly in $[-\pi, \pi]$. Moreover, if $\Phi(x)$ is an indefinite integral of $F(x)$, then

$$\Phi(x) = \frac{a_0}{4} x^2 + Cx + D - \sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^2}$$

and therefore the Riemann function $\Phi(x)$ for the series (69.1) is the result of the double successive integration of $f(x)$. But since $F(x)$ is continuous, then $\Phi'(x) = F(x)$ at every point; moreover $F'(x) = f(x)$ almost everywhere; thus $\Phi''(x) = f(x)$ almost everywhere but since $D^2\Phi(x) = \Phi''(x)$ then, where $\Phi''(x)$ exists (§ 67), $D^2\Phi(x) = f(x)$ almost everywhere, and therefore the series (69.1) is summable almost everywhere to $f(x)$ by Riemann's method.

The theorem is proved.

We now begin to apply Riemann's method to general trigonometric series and especially to the very important question of the uniqueness of the expansion of a function into a trigonometric series.

§ 70. Cantor's theorem of uniqueness

Using Riemann's method of summation, we can answer the following important question; can two different trigonometric series exist which converge at every point to the same function $f(x)$? The answer to this question is in the negative. In order to prove this, we first prove the following important theorem:

CANTOR'S[1] THEOREM. *If the trigonometric series*

$$\frac{a_0}{2} + \sum (a_n \cos nx + b_n \sin nx) \tag{70.1}$$

converges to zero at every point x of $[0, 2\pi]$, *then all its coefficients equal zero.*

According to Cantor's Theorem, the coefficients of the series (70.1) tend to zero (this follows not from the Cantor–Lebesgue theorem, but from Cantor's own theorem — see § 62, note). If we construct the Riemann function $F(x)$ for series (70.1), it is continuous along the whole infinite straight line. According to the theorem in § 68, the series (70.1) should be summable to zero at every point, i.e.

$$D^2F(x) = 0 \quad -\pi \leqslant x \leqslant \pi.$$

Then according to the theorem of § 67 we have

$$F(x) = Ax + B. \tag{70.2}$$

But on the other hand since $F(x)$ is the Riemann function for the series (70.1), then

$$F(x) = \frac{a_0}{4}x^2 + Cx + D - \sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^2}. \tag{70.3}$$

From (70.2) and (70.3) we obtain

$$\frac{a_0}{4}x^2 + A_1 x + B_1 = \sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^2}, \tag{70.4}$$

where A_1 and B_1 are new constants. But the right-hand side of (70.4) has a period 2π, which means that the same applies to the left-hand side and this is possible only for

$$a_0 = 0 \quad \text{and} \quad A_1 = 0. \tag{70.5}$$

We now have

$$B_1 = \sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^2}. \tag{70.6}$$

Series (70.6) converges uniformly; therefore (see § 12) its coefficients are the Fourier coefficients for its sum, but that is a constant number B_1 and therefore

$$\frac{a_n}{n^2} = \frac{b_n}{n^2} = 0 \quad (n = 1, 2, \ldots),$$

whence

$$a_n = b_n = 0 \quad (n = 1, 2, \ldots). \tag{70.7}$$

From (70.5) and (70.7) it follows that series (70.6) has all its coefficients equal to zero and thus Cantor's theorem is proved. He immediately generalized this theorem, by proving the following statement:

If a trigonometric series converges to zero everywhere apart, perhaps, from a finite number of points, then all its coefficients equal zero.

In fact, arguing in exactly the same way as in the proof of the preceding theorem, we see that the series under consideration has coefficients tending to zero and its Riemann function $F(x)$ should be linear in every interval where the series converges to zero, since then $D^2 F(x) \equiv 0$. But $F(x)$ should be smooth by virtue of Theorem 3 of § 68. Therefore it cannot possess angular points. Consequently it cannot consist of different rectilinear intervals and should be simply linear. But if this is so, then the proof is concluded as in the previous theorem, i.e. we prove that all the coefficients of the series equal zero.

Note. Cantor's theorem can be expressed in the following more general form: *if a trigonometric series with coefficients tending to zero is summable to zero by Riemann's method everywhere apart, perhaps, from a finite number of points, then all its coefficients equal zero.*

Indeed, in proving the theorem we only use the facts that the coefficients of the series tend to zero and $D^2 F(x) = 0$ everywhere apart, perhaps, from a finite number of points.

COROLLARY. *Let $f(x)$ be a function with period 2π, which is finite at every point of* $[0, 2\pi]$. *Then, there do not exist two different trigonometric series, each of which converges to $f(x)$ everywhere in* $[0, 2\pi]$ *apart, perhaps, from a finite number of points.*

Indeed, we will suppose that two such trigonometric series do exist; then their difference would be the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \tag{70.8}$$

in which not all the coefficients equal zero, but it converges to zero everywhere apart, perhaps, from a finite number of points. However, we have already seen that this is impossible.

Here, it is true, the requirement of convergence can be replaced by summability by the Riemann method (but in this case it is previously required that the coefficients tend to zero).

The theorem on the uniqueness of the expansion of a function into a trigonometric series permits considerable generalizations. We will devote Chapter XIV to this problem; here we will confine ourselves to formulating the most important results. For this purpose we introduce a definition.

DEFINITION. The set E, lying in $[-\pi, \pi]$, is known as *an M-set,* if there exists a trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

in which not all the coefficients equal zero and which converges to zero everywhere in $[-\pi, \pi]$ outside the set E.

If the set E is not an M-set, then we call it a U-set†.

Using this definition, we can now formulate the two preceding theorems thus; *if E is an empty or finite set, then it is a U-set.*

Cantor himself proved that any reducible set (i.e. one for which the derived set is finite or denumerable) is again a U-set. Subsequently, Young[1] proved that any denumerable set is a U-set (see § 5, Chapter XIV).

On the other hand, it is easily proved that *any set E, $mE > 0$, is an M-set.* In fact let us take a perfect set $P \in E$, $mP > 0$, and suppose that $f(x) = 1$ in P and $f(x) = 0$ outside P. From the principle of localization (see § 33) the series $\sigma(f)$ converges to zero in every interval adjoining P and therefore everywhere outside E. Thus there exists a trigonometric series convergent to zero everywhere outside P but with coefficients differing from zero (for example

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx = \frac{1}{\pi} mP).$$

Consequently, E is an M-set.

For a long time it was supposed that, on the contrary, any set of measure zero (not only finite and denumerable) should be a U-set. This hypothesis was refuted by Men'shov[1], who set up the first example of a perfect M-set of measure zero (see the proof in § 12, Chapter XIV).

§ 71. Riemann's principle of localization for general trigonometric series

The function $F(x)$ introduced by Riemann plays an important role not only in the question of the uniqueness of the expansion of a function into a trigonometric series but also in the examination of its convergence or divergence.

We recall that the following theorem was proved for Fourier series (see § 33): the convergence or divergence of a series $\sigma(f)$ at a point x depends only on the behaviour of the function $f(x)$ in the neighbourhood of the point x.

We will now suppose that we are concerned with an arbitrary trigonometric series, not a Fourier series. It seems that it is then possible to judge its convergence by

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† From the definition, it immediately follows that any part of a U-set is a U-set: on the other hand, a set containing an M-set is itself an M-set.

studying its Riemann function. Thus we have a theorem, analogous to the preceding theorem, which can be expressed in this form:

For any trigonometric series with coefficients tending to zero, the convergence or divergence of the series at some point x depends only on the behaviour of the Riemann function $F(x)$ in the neighbourhood of the point x.

This somewhat indistinct formulation will be stated more exactly later (see p. 200). Riemann proved this statement thus: he constructed a function $\lambda(x)$, equal to unity in $[\alpha, \beta]$, equal to zero outside (a, b) and possessing continuous derivatives up to the fourth order inclusive in $[0, 2\pi]$. After this, he proved that the difference

$$\frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) - \frac{1}{\pi} \int_a^b F(t)\, \lambda(t)\, \frac{d^2}{dt^2} D_n(t - x)\, dt \qquad (71.1)$$

tends to zero uniformly in $[\alpha, \beta]$ and from this he drew the necessary conclusion.

At the present time the idea of introducing a function $\lambda(x)$ has been completely maintained but the proof of Riemann's theorem is usually carried out using the theory of the *formal multiplication of series*†; by the way, this theory also gives many other useful results which we will prove in Chapter XIV.

Thus, we begin with the concept of the formal product of two trigonometric series. For simplicity of exposition we will write the trigonometric series in its complex form

$$\sum_{n=-\infty}^{n=+\infty} c_n e^{inx} \qquad (c_{-n} = \bar{c}_n).$$

Let us consider two trigonometric series

$$\sum_{n=-\infty}^{n=+\infty} c_n e^{inx} \qquad (71.2)$$

and

$$\sum_{n=-\infty}^{n=+\infty} \gamma_n e^{inx}. \qquad (71.3)$$

Let us call their *formal product* the series

$$\sum_{n=-\infty}^{n=+\infty} K_n e^{inx}, \qquad (71.4)$$

where

$$K_n = \sum_{p=-\infty}^{p=+\infty} c_p \gamma_{n-p} \qquad (71.5)$$

on the supposition that all the series (71.5), defining K_n, converge ($n = 0, \pm 1, \pm 2, \ldots$).

In all that follows we will be concerned with the case when $\sum |\gamma_n| < +\infty$. Under these conditions series (71.3) converges absolutely and uniformly in $[-\pi, \pi]$ and is

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† This theorem is due to Rajchman (see Rajchman[1] and also Zygmund[12]).

the Fourier series of some function $\lambda(x)$. As regards the series (71.2), it can be any series† providing that

$$c_n \to 0 \quad \text{as} \quad n \to \pm\infty.$$

Let us prove the following two lemmas due to Rajchman.

LEMMA 1. *If $c_n \to 0$ as $n \to \pm\infty$ and if the series $\sum |\gamma_n|$ converges, then all K_n defined by formula* (71.5) *have meaning and $K_n \to 0$ as $n \to \pm\infty$.*

Indeed, if $M = \max |c_n|$, then as $n \to +\infty$

$$|K_n| \leqslant M \sum_{p=-\infty}^{\left[\frac{n}{2}\right]} |\gamma_{n-p}| + \max_{p<\left[\frac{n}{2}\right]} |c_p| \sum_{p=\left[\frac{n}{2}\right]+1}^{\infty} |\gamma_{n-p}|$$

$$\leqslant M \sum_{q=n-\left[\frac{n}{2}\right]}^{\infty} |\gamma_q| + \max_{p>\left[\frac{n}{2}\right]} |c_p| \sum_{q=-\infty}^{q=+\infty} |\gamma_q| \to 0 \quad \text{as} \quad n \to +\infty$$

and similarly we can carry out the proof for $n \to -\infty$. Lemma 1 is proved.

It is said that series (71.3) *converges rapidly* to S, if it converges to S and if the series

$$\Gamma_0 + \Gamma_1 + \cdots + \Gamma_n + \cdots,$$

converges, where

$$\Gamma_n = \sum_{k=n}^{\infty} |\gamma_k|.$$

Thus, for example, if the coefficients of series (71.3) are of order $0(1/n^3)$, then $\Gamma_n = 0(1/n^2)$ and, therefore, series (71.3) converges rapidly. Subsequently we will frequently use the series $\sigma(\lambda)$ for the series (71.3), where $\lambda(x)$ is a function possessing three continuous derivatives. Then the coefficients of the series $\sigma(\lambda)$ will be of order $0(1/n^3)$ (see § 24) and $\sigma(\lambda)$ will converge rapidly to $\lambda(x)$.

We will now turn to proving the following lemma.

LEMMA 2. *If $c_n \to 0$ as $n \to \pm\infty$ and series* (71.3) *converges rapidly to zero in some set E, then the formal product* (71.4) *converges to zero uniformly in the set E.*

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† It is appropriate to note here that if series (71.2) is the Fourier series of some function $f(x)$, then the formal product becomes the Fourier series of $f(x)\,\lambda(x)$. Indeed, if we denote by K_n the Fourier coefficients of $f(x)\,\lambda(x)$, then

$$K_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, e^{-int} \lambda(t)\, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, e^{-int} \sum_{q=-\infty}^{q=+\infty} \gamma_q e^{iqt}\, dt$$

$$= \sum_{q=-\infty}^{q=+\infty} \gamma_q \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, e^{-i(n-q)t}\, dt = \sum_{q=-\infty}^{q=+\infty} c_{n-q}\gamma_q = \sum_{p=-\infty}^{p=+\infty} c_p \gamma_{n-p}.$$

Here term-by-term integration was valid, since we had assumed that $\sum |\gamma_n| < +\infty$, and therefore series $\sigma(\lambda)$ converges uniformly.

Indeed, let $x_0 \in E$ and

$$R_k(x) = \sum_{n=k}^{\infty} \gamma_n e^{inx}.$$

We have for $k > 0$

$$|R_{-k}(x_0)| = \left|\sum_{-k}^{\infty} \gamma_n e^{inx_0}\right| = \left|\sum_{-(k+1)}^{-\infty} \gamma_n e^{inx_0}\right| = |R_{k+1}(x_0)| \leqslant \Gamma_{k+1} \tag{71.6}$$

which means that the series

$$\sum_{k=-\infty}^{k=+\infty} |R_k(x)|$$

converges uniformly for $x \in E$.

Then

$$Q_m(x_0) = \sum_{n=-m}^{n=+m} K_n e^{inx_0} = \sum_{n=-m}^{n=+m} e^{inx_0} \sum_{p=-\infty}^{p=+\infty} c_p \gamma_{n-p} = \sum_{p=-\infty}^{p=+\infty} c_p e^{ipx_0} \sum_{n=-m}^{n=+m} \gamma_{n-p} e^{i(n-p)x_0}$$

$$= \sum_{p=-\infty}^{p=+\infty} c_p e^{ipx_0} \sum_{q=-m-p}^{m-p} \gamma_q e^{iqx_0}$$

$$= \sum_{p=-\infty}^{p=+\infty} c_p e^{ipx_0} R_{-m-p}(x_0) - \sum_{p=-\infty}^{p=+\infty} c_p e^{ipx_0} R_{m-p+1}(x_0).$$

Therefore

$$|Q_m(x_0)| \leqslant \sum_{p=-\infty}^{p=+\infty} |c_p|\,|R_{-m-p}(x_0)| + \sum_{p=-\infty}^{p=+\infty} |c_p|\,|R_{m-p-1}(x_0)|,$$

and, taking into account the inequality (71.6), by the same arguments as in Lemma 1, we prove that $Q_m(x_0) \to 0$ as $m \to \infty$ and moreover uniformly for $x_0 \in E$, since the estimate of $R_k(x_0)$ in terms of Γ_k or Γ_{k+1} is valid for all $x \in E$.

From these two lemmas we can deduce a theorem:

THEOREM 1. *If the series* (71.3) *converges rapidly to some function* $\lambda(x)$ *and* $c_n \to 0$, *then the series*

$$\sum_{n=-\infty}^{n=+\infty} [K_n - \lambda(x)\, c_n]\, e^{inx} = \sum_{n=-\infty}^{n=+\infty} K_n e^{inx} - \lambda(x) \sum_{n=-\infty}^{n=+\infty} c_n e^{inx} \tag{71.7}$$

converges uniformly to zero in $[-\pi, \pi]$.

In order to prove this we suppose that

$$\gamma_0^* = \gamma_0 - \lambda(x),$$

$$\gamma_n^* = \gamma_n \qquad \text{for} \quad n \neq 0$$

and set up the formal product $\sum K_n^* e^{inx}$ of the series $\sum c_n e^{inx}$ and $\sum \gamma_n^* e^{inx}$. It is true that in the latter series γ_0^* is not a constant value, but it is not difficult to show that the proof of Lemma 2 would not be changed, if we supposed γ_0 to be a bounded function of x, which occurs in our example. Therefore we can apply Lemma 2, since the series $\sum \gamma_n^* e^{inx}$ converges rapidly to zero in $[-\pi, \pi]$ and we find that $\sum K_n^* e^{inx}$ converges rapidly to zero uniformly in $[-\pi, \pi]$. But

$$K_n^* = \sum_{p=-\infty}^{p=+\infty} c_p \gamma_{n-p}^* = c_n[\gamma_0 - \lambda(x)] + \sum_{p \neq n}{}' c_p \gamma_{n-p} = K_n - \lambda(x)\, c_n,$$

and therefore

$$\sum K_n e^{inx} - \lambda(x) \sum c_n e^{inx}$$

converges to zero uniformly in $[-\pi, \pi]$ and Theorem 1 is proved.

Combining Lemma 2 and Theorem 1, we can express a proposition which we will use later.

COROLLARY 1. *Let $\lambda(x)$ be a function for which the Fourier series converges rapidly and $\sum c_n e^{inx}$ a series whose coefficients $c_n \to 0$ as $n \to \pm\infty$. Then the formal product of the series $\sum c_n e^{inx}$ and the Fourier series for $\lambda(x)$ converges to zero everywhere where $\lambda(x) = 0$ (even if the series $\sum c_n e^{inx}$ diverges). At those points where $\lambda(x) \neq 0$, it diverges if the series $\sum c_n e^{inx}$ diverges and it converges to $\lambda(x_0)\, S(x_0)$ if $\sum c_n e^{inx}$ converges to $S(x_0)$.*

Note 1. We will remark that this statement can be strengthened. Thus, if we suppose that $\lambda(x) \neq 0$ at some point then

$$\left.\begin{aligned} \overline{\lim}\, Q_n(x_0) &= \lambda(x_0)\, \overline{\lim}\, S_n(x_0), \\ \underline{\lim}\, Q_n(x_0) &= \lambda(x_0)\, \underline{\lim}\, S_n(x_0) \end{aligned}\right\} \quad \text{at} \quad \lambda(x_0) > 0$$

and

$$\left.\begin{aligned} \overline{\lim}\, Q_n(x_0) &= \lambda(x_0)\, \overline{\lim}\, S_n(x_0), \\ \underline{\lim}\, Q_n(x_0) &= \lambda(x_0)\, \underline{\lim}\, S_n(x_0) \end{aligned}\right\} \quad \text{at} \quad \lambda(x_0) < 0.$$

This follows immediately from an examination of the partial sums of the series

$$\sum K_n e^{inx} - \lambda(x) \sum c_n e^{inx},$$

which, as we have seen, converges to zero. Therefore, in particular, if $\overline{\lim}\, |S_n(x_0)| = +\infty$, then $\overline{\lim}\, |Q_n(x_0)| = +\infty$ also.

This result will be used specifically in Chapter XIV.

From Theorem 1 it also follows immediately that:

COROLLARY 2. *If the Fourier series for $\lambda(x)$ converges rapidly and $\sum c_n e^{inx}$ converges uniformly in E to $S(x)$, then the formal product converges uniformly in E to $\lambda(x)\, S(x)$. If in a set F we have $|\lambda(x)| > a > 0$, then the uniform convergence of a formal product in E implies the uniform convergence of $\sum c_n e^{inx}$ in it.*

Note 2. In Corollaries 1 and 2 the words "convergence" or "uniform convergence" can be replaced by "summability" or "uniform summability" by the Riemann method. In fact, according to Theorem 1,

$$\sum_{-\infty}^{+\infty} [K_n - \lambda(x)\, c_n]\, e^{inx}$$

converges to zero uniformly in $[-\pi, \pi]$. By virtue of the note to Theorem 2 of § 68 it follows that this series is uniformly summable to zero by the Riemann method in $[-\pi, \pi]$, and this means that the difference of the series

$$\sum K_n e^{inx} \quad \text{and} \quad \lambda(x) \sum c_n e^{inx}$$

is uniformly summable to zero by the Riemann method in $[-\pi, \pi]$, whence the required result immediately follows.

We can now prove the following important theorem:

THEOREM 2. *If a trigonometric series with coefficients tending to zero is summable by the Riemann method to zero at every point of some interval (a, b), then it converges to zero at every point of (a, b) and moreover uniformly in any interval lying entirely within (a, b).*

Let $\lambda(x) = 1$ in $[\alpha, \beta]$, $\lambda(x) = 0$ outside (a, b) and $\lambda(x)$ be interpolated between $[a, \alpha]$ and $[b, \beta]$ as desired, provided it possesses continuous derivatives up to the

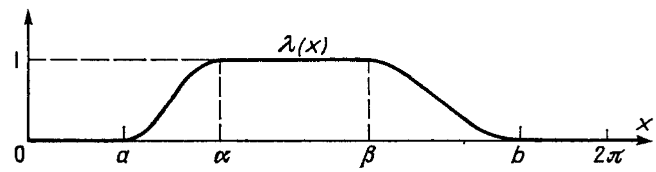

FIG. 14

hird order inclusive (see Fig. 14). We have already said that under these conditions he series $\sigma(\lambda)$ converges rapidly. Let

$$\sigma(\lambda) = \sum \gamma_n e^{inx}.$$

We will set up the formal product (71.4) of the given series and the series $\sigma(\lambda)$. Since $\lambda(x) = 0$ outside (a, b), then by virtue of Corollary 1, the series (71.4) converges to zero outside (a, b), which means that it is summable outside (a, b) to zero by the Riemann method. Moreover, by virtue of Corollary 1 and Note 2 concerning summability, series (71.4) is summable to zero by the Riemann method at every point of (a, b), because we have assumed that this holds for series (71.2) in (a, b). Thus, series (71.4) is summable by the Riemann method to zero at every point in $[-\pi, \pi]$. If this is so, then according to the theorem of § 70 (see its note), it possesses all coefficients equal to zero. But according to Theorem 1 of this section the series

$$\sum K_n e^{inx} - \lambda(x) \sum c_n e^{inx}$$

converges to zero uniformly in $[-\pi, \pi]$. If all $k_n = 0$, then this means that

$$\lambda(x) \sum c_n e^{inx}$$

converges uniformly to zero in $[-\pi, \pi]$. But $\lambda(x) = 1$ in $[\alpha, \beta]$, therefore $\sum c_n e^{inx}$ converges uniformly to zero in $[\alpha, \beta]$, and the proof of Theorem 2 is concluded.

Now we will express in an exact form and prove a theorem which was formulated to some extent at the beginning of this section. So we have the following theorem which is known as Riemann's principle of localization.

RIEMANN'S PRINCIPLE OF LOCALIZATION. *Let $F_1(x)$ and $F_2(x)$ be Riemann functions for two trigonometric series with coefficients tending to zero; if these functions are equal in some interval (a, b) or, perhaps, if their difference is a linear function in (a, b), then the difference of the given trigonometric series is a series convergent to zero everywhere in (a, b) and moreover, uniformly in any interval $[\alpha, \beta]$ lying entirely within (a, b).*

To prove the theorem, let us consider two trigonometric series with coefficients tending to zero. Let (71.2) be the difference of these series, and $F_1(x)$ and $F_2(x)$ their Riemann functions. Then, according to the condition of the Riemann theorem, the sum $F(x)$ for series (71.2) is a linear function in (a, b). If this is so,

$$D^2 F(x) = 0 \quad \text{in} \quad (a, b)$$

then, consequently, series (71. 2) is summable to zero by the Riemann method at every point of the interval (a, b) and Theorem 2 can be applied.

From Riemann's principle of localization the truth of the statement made at the beginning of the section follows immediately, the convergence or divergence of a series with coefficients tending to zero depends only on the behaviour of the Riemann function.

Indeed, if for two series with coefficients tending to zero, we have $F_1(x) = F_2(x)$ in (a, b), then the convergence or divergence of both series at any point $x \in (a, b)$ can only occur simultaneously (and also if they converge, they possess the same sum). It is in this sense that it should be understood that convergence or divergence depends only on the behaviour of the Riemann function.

It should be noted that the general Riemann principle of localization proved here includes, as a particular case, Riemann's principle of localization for Fourier series (see § 33). Indeed, if the two given series are Fourier series for $f_1(x)$ and $f_2(x)$, then the functions $F_1(x)$ and $F_2(x)$ are obtained as a result of the double successive integration of $f_1(x)$ and $f_2(x)$ (see § 70), and, therefore, if $f_1(x) = f_2(x)$ in (a, b), then $F_1(x) - F_2(x)$ will be linear in this interval, and if the general principle of localization has already been proved, then it can be stated that $\sigma(f_1) - \sigma(f_2)$ converges to zero in (a, b) everywhere and moreover uniformly in $[\alpha, \beta]$, lying within (a, b).

In Chapter XIV we shall see the part played by Riemann's principle of localization which has been established here.

§ 72. du Bois-Reymond's theorem

Let $f(x)$ be a function which is finite at every point of $[-\pi, \pi]$. We have already seen (see § 70) that there cannot exist two different trigonometric series converging to it everywhere in $[-\pi, \pi]$. But if one such series exists, ought it to be its Fourier series?

This question, of course, only has meaning for summable $f(x)$, since otherwise it would be simply impossible to write down the Fourier series (we always mean Fourier–Lebesgue series).

Let us note that the convergence of a trigonometric series at every point does not in any way imply that it is a Fourier series. Indeed, for example, the series

$$\sum_{n=2}^{\infty} \frac{\sin nx}{\ln n}$$

converges everywhere, since this is a sine series with monotonically decreasing coefficients (see § 30); however, it is not a Fourier series (see § 40).

Therefore it is appropriate to pose the question thus: Let $f(x)$ be finite at every point and summable. Let there exist a trigonometric series converging to it everywhere in $[-\pi, \pi]$. Could this series be its Fourier series?

Here we can give a positive answer to this question in the case when $f(x)$ is a bounded function; it was in this form that the theorem was proved by Lebesgue who generalized the initial result obtained by du Bois-Reymond.† But before proving this theorem, we should demonstrate the validity of the following lemma:

LEMMA. *If $F(x)$ is continuous in $[a, b]$ and*

$$m \leqslant D^2 F(x) \leqslant M \quad \text{in} \quad (a, b),$$

then for any x_0 and h such that $a \leqslant x_0 - 2h < x_0 + 2h \leqslant b$, we have

$$m \leqslant \frac{F(x_0 + 2h) + F(x_0 - 2h) - 2F(x_0)}{4h^2} \leqslant M.$$

To prove this, we consider the auxiliary function

$$\Psi(x) = F(x_0) + (x - x_0)\frac{F(x_0 + 2h) - F(x_0 - 2h)}{4h}$$
$$+ \frac{(x - x_0)^2}{2} \frac{F(x_0 + 2h) + F(x_0 - 2h) - 2F(x_0)}{4h^2}.$$

It is clear that $\Psi(x)$ is a polynomial of the second degree in x, whilst

$$\Psi(x_0 + 2h) = F(x_0 + 2h),\ \Psi(x_0) = F(x_0) \quad \text{and} \quad \Psi(x_0 - 2h) = F(x_0 - 2h),$$

i.e., the difference

$$r(x) = F(x) - \Psi(x)$$

becomes zero at $x = x_0 - 2h$, x_0 and $x_0 + 2h$. Moreover, $r(x)$ is continuous in $[a, b]$ and

$$D^2 r(x) = D^2 F(x) - \frac{F(x_0 + 2h) + F(x_0 - 2h) - 2F(x_0)}{4h^2}.$$

Since $r(x)$ possesses a minimum and maximum somewhere inside $(x_0 - 2h, x_0 + 2h)$, let them be the points x_1 and x_2, and at them it is known that $D^2 r(x_1) \geqslant 0$ and $D^2 r(x_2) \leqslant 0$, so it is immediately clear that

$$D^2 F(x_2) \leqslant \frac{F(x_0 + 2h) + F(x_0 - 2h) - 2F(x_0)}{4h^2} \leqslant D^2 F(x_1),$$

which proves the validity of the lemma.

We can now prove the theorem:

THE DU BOIS-REYMOND–LEBESGUE THEOREM: *If $f(x)$ is bounded in $[-\pi, \pi]$ and there exists a trigonometric series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \tag{72.1}$$

converging to it everywhere in this interval, then this series is its Fourier series.

† du Bois-Reymond[2] considered only the case of bounded functions, integrable in the Riemann sense.

We will first note that from the convergence of series (72.1) it follows that $a_n \to 0$ and $b_n \to 0$ (see § 62). Therefore, it is possible to construct a Riemann function and to obtain, as in § 68

$$\frac{F(x_0+2h)+F(x_0-2h)-2F(x_0)}{4h^2}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx_0 + b_n \sin nx_0) \left(\frac{\sin nh}{nh}\right)^2. \tag{72.2}$$

From Riemann's theorem (see § 68, Theorem 1) we have at every point

$$D^2 F(x) = f(x). \tag{72.3}$$

But $f(x)$ is given as bounded; which means that by the preceding lemma

$$\left|\frac{F(x+2h)+F(x-2h)-2F(x)}{4h^2}\right| \leqslant M, \tag{72.4}$$

where M is a constant (and this is for any h and any x, $-\pi \leqslant x \leqslant \pi$). We also note that $f(x)$, as the sum of an everywhere convergent series of continuous functions, is measurable, which means that being measurable and bounded, it is summable.

From the uniform convergence of (72.2) it follows that it is the Fourier series of the function on the left-hand side of the equality, i.e.

$$a_n \left(\frac{\sin nh}{nh}\right)^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{F(x+2h)+F(x-2h)-2F(x)}{4h^2} \cos nx \, dx \tag{72.5}$$

and similarly

$$b_n \left(\frac{\sin nh}{nh}\right)^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{F(x+2h)+F(x-2h)-2F(x)}{4h^2} \sin nx \, dx. \tag{72.6}$$

But

$$D^2 F(x) = \lim_{h \to 0} \frac{F(x+2h)+F(x-2h)-2F(x)}{4h^2}.$$

Therefore by virtue of (72.3)

$$\lim_{h \to 0} \frac{F(x+2h)+F(x-2h)-2F(x)}{4h^2} = f(x).$$

If we now note that due to (72.4) the expressions under the integral sign in the integrals (72.5) and (72.6) are bounded at any x and h by the same value M (this is true for any n), then it is possible to carry out the passage to the limit under the integral sign, and therefore

$$a_n = \lim_{h\to 0} a_n \left(\frac{\sin nh}{nh}\right)^2$$

$$= \lim_{h\to 0} \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{F(x+2h) + F(x-2h) - 2F(x)}{4h^2} \cos nx \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

and similarly

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx,$$

and this is what it was required to be proved.

Note. We assumed $f(x)$ to be bounded but the given theorem permits considerable generalization. It need not require convergence of the series at every point of $[0, 2\pi]$ (see Chapter XIV, § 4).

§ 73. Problems

1. The series

$$\sum_{n=1}^{\infty} \cos nx \quad \text{and} \quad \sum_{n=1}^{\infty} \cos 2^n x$$

do not possess points of convergence, if the series

$$\text{(a)} \quad \sum_{n=1}^{\infty} \sin nx$$

converges only at $x \equiv 0 \pmod{\pi}$, whilst the series

$$\text{(b)} \quad \sum_{n=1}^{\infty} \sin 2^n x$$

possesses an infinite (but denumerable) set of points of convergence in $(0, \pi)$.

The set of points of normal convergence of series (b) coincides with the set of points of its absolute convergence.

[In considering series (b), represent the points x in the form $x = \pi y = \pi \sum_{n=1}^{\infty} \delta_k/2^k$ where $\delta_k = 0$ or 1. Consider the cases when y is a binary rational number and when y is not a binary rational number.]

2. The set $E \subset (-\infty, +\infty)$ of all the points of convergence of the series

$$\sum_{n=1}^{\infty} n \sin 2^{n!} x$$

has the power of the continuum in any interval (a, b) where $a < b$ (although $mE = 0$).

This is also true for the series

$$\sum_{n=1}^{\infty} n \cos 2^{n!} x.$$

[Consider the set of points $E = \prod_{n=10}^{\infty} E_n$, where $E_n = \{-\infty < x < \infty : |\sin 2^{n!} x \leqq 1/n^3\}$.]

3. If the trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{73.1}$$

converges in measure in some set E where $mE > 0$, then

$$\lim_{n\to\infty} \sqrt{a_n^2 + b_n^2} = 0.$$

[Refer to the proof of the Cantor–Lebesgue theorem in § 62.]

4. Consider a measurable function $\varphi(x)$ which is 2π-periodic and not equal to a constant. Then for any α_n and λ_n where $(\lambda_n) \to \infty$ as $n \to \infty$, the sequence of functions $\{\varphi(\lambda_n x - \alpha_n)\}$ diverges almost everywhere in $(-\infty, \infty)$.

5. If functions $g(x) \in C(0, 2\pi)$ and $f(x) \in C(0, 2\pi)$ exist such that $g(x) = f(x)$ for $x \in [1, 2]$, the Fourier series of $g(x)$ and $f(x)$ are however not uniformly equiconvergent in the interval $(1, 2)$.

6. The absolute convergence of trigonometric series is not a local property. There exists a 2π-periodic absolutely-continuous function $f(x)$, the Fourier series of which is not absolutely convergent at any point $x \in [1, 2]$, although $f(x) = 0$ for $x \in [1, 2]$.

7. There exists a trigonometric series of the the form (73.1) which diverges everywhere in $(-\infty, \infty)$ and get this series is summable by the Abel–Poisson method for all $x \in (-\infty, \infty)$.

[Take the series $\sum_{n=1}^{\infty} n \sin nx$ and add to it a Fourier series (of a continuous function) which diverges only at the points $x \equiv O \pmod{\pi}$.]

8. Let $a_n \downarrow 0$ and

$$f(x) = \sum_{n=1}^{\infty} a_n \sin nx. \tag{73.2}$$

Then

(a) if $\lim_{x\to+0} f(x) = 0$, $a_n = o(1/n)$ and the series (73.2) converges uniformly in $[0, \pi]$;

(b) if $\overline{\lim_{x\to+0}} f(x) = A$, where A is a finite number, $a_n = O(1/n)$ and

$$\left|\sum_{n=1}^{k} a_n \sin nx\right| \leqslant D \text{ for all } \begin{cases} k = 1, 2 \ldots \\ x \in [0, \pi] \end{cases}$$

where D is a finite number.

9. Let

$$a_n \downarrow 0$$

and

$$f(x) = \sum_{n=1}^{\infty} a_n \cos nx. \tag{73.3}$$

Then

(a) if $\lim\limits_{x\to+0} f(x) = A$, where A is a finite number,

$$A = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n$$

and the series (73.3) converges absolutely in $(-\infty, \infty)$;

(b) if $\lim\limits_{x\to+0} f(x) = +\infty$, $f(x) \in L(0, 2\pi)$ and the series (73.3) is a Fourier series;

(c) if the function $f(x)$ is non-integrable in $[0, \pi]$, then

$$-\infty = \underline{\lim}_{x\to+0} f(x) < \overline{\lim}_{x\to+0} f(x) = +\infty.$$

10. (i) If a 2π-periodic function $f(x) \in \mathrm{Lip}\,\alpha$, then

$$\|f(x) - \sigma_n(x,f)\|_c = \begin{cases} O(1/n^\alpha) & \text{for } 0 < \alpha < 1 \\ O((\log n)/n) & \text{for } \alpha = 1, \end{cases} \tag{73.4}$$

where $\sigma_n(x,f)$ are the arithmetic means of the partial sums of the Fourier series of the function f. The estimate (73.4) cannot be bettered.

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[The estimate (73.4) follows from the fact that (see § 47)

$$\|f - \sigma_n\|_c \leqslant \frac{1}{\pi}\left\| \int_0^{1/n} |f(x+t) + f(x-t) - 2f(x)|\, K_n(t)\, dt \right\|$$
$$+ \frac{8}{\pi(n+1)} \int_{1/n}^{\pi} \frac{\|f(x+t) + f(x-t) - 2f(x)\|}{t^2}\, dt.$$

If we take the function $f_0(x) \in \mathrm{Lip}\,\alpha$ such that $f_0(x) = |x|^\alpha$ where $|x| \leqslant 1$, then we prove that the estimate (73.4) cannot be bettered with respect to order.]

(ii) If $f(x) \in C(0, 2\pi)$, then

$$\|f(x) - \sigma_n(x,f)\|_c = O\left\{\frac{1}{n}\sum_{k=0}^{n} E_k(f)\right\}.$$

S. B. STECHKIN

11. Consider a 2π-periodic function $f(x) = x\sin(\pi/x)$ where $0 < x < 1$ and $f(x) = 0$ for $1 \leqslant x \leqslant 2\pi$. Then

$$\|f(x) - \sigma_n(x,f)\|_c = O(1/\sqrt{n})$$

and this estimate cannot be bettered with respect to order.

[The modulus of continuity $\omega(\delta, f) = O(\delta^{1/2})$ and $\omega(\delta, f) \neq o(\delta^{1/2})$. The fact that the estimate cannot be bettered follows from Stechkin's result[5] (see also § 7 of the Appendix).]

12. For every $\alpha \in (0, 1)$ there exists a function $f \in \mathrm{Lip}\,\alpha$ such that

$$\lim_{n\to\infty} n^\alpha |f(x) - \sigma_n(x,f)| > 0 \text{ for nearly all } x \in (-\infty, +\infty).$$

For $\alpha = 1$, a statement of this type is not valid.

A. I. RUBINSHTEIN

13. Consider a 2π-periodic function

$$f(x) = \begin{cases} 1/q & \text{for} \quad x = 2\pi p/q \\ 0 & \text{for} \quad \text{the remaining } x \text{ in } [0, 2\pi], \end{cases}$$

where $0 < p < q$ and p/q is an irreducible fraction. Then there does not exist a trigonometric series which would converge everywhere to $f(x)$.

[Apply the du Bois-Reymond–Lebesgue Theorem of § 72.]

14. Construct some measurable set E in $(0, 2\pi)$ such that for any interval $(a, b) \subset (0, 2\pi)$ the measures $m(a, b)\, E > 0$ and $m(a, b)\, CE > 0$ where $CE = [0, 2\pi] - E$.

15. Consider the E-set of problem 14. Then, if

$$f(x) = \begin{cases} 1 & \text{for} \quad x \in E \\ 0 & \text{for} \quad x \in CE \end{cases}$$

there does not exist a trigonometric series which would converge to $f(x)$ in some interval (a, b).

[Assume the opposite and apply Baire's theorem concerning the limit of a sequence of continuous functions (see Lusin, A. 17, § 47).]

16. There does not exist a denumerable system of functions $f_n(x) \in C(0, 1)$ such that the set of all the functions

$$F(x) \equiv F(x; \{c_k\}, N) = \sum_{k=1}^{N} c_k f_k(x) \qquad (N = 1, 2, \ldots)$$

(where c_k are arbitrary real numbers) coincides with the whole space $C(0, 1)$.

[Assume the opposite and consider the function

$$\varphi(x) = \sum_{n=1}^{\infty} \frac{\sqrt{\omega(x, f_n)}}{(A_n + 1)\, 2^n} \quad (x \in [0, 1]),$$

where $A_n = \sup_{x \in [0,1]} |f_n(x)|$ and $\omega(x, f)$ is the modulus of continuity of f.]

17. If

$$\sum_{k=1}^{\infty} |a_k \cos \sqrt{2}\pi k| < \infty,$$

then

$$\sum_{k=1}^{\infty} \frac{|a_k|}{k} < \infty.$$

A. A. Muromskii

18. (i) If there exists a set $E \subset [0, 1]$ with $mE = 1$ such that if

$$\sum_{k=1}^{\infty} |a_k \cos \pi k x_0| < \infty$$

at some point $x_0 \in E$, then

$$\sum_{k=2}^{\infty} \frac{|a_k|}{k (\ln k)^{1+\varepsilon}} < \infty$$

for any $\varepsilon > 0$.

(ii) Prove that the set E (of part (i)) can be chosen such that it contains all the algebraic irrational points of the interval $(0, 1)$.

A. A. Muromskii

19. (i) If the series

$$\sum_{n=1}^{\infty} c_n(1 - \cos nx) \tag{73.5}$$

converges at every point of an interval (a, b), then the series

$$\sum_{n=1}^{\infty} c_n \tag{73.6}$$

converges.

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(ii) If $c_n = \dfrac{(-1)^n}{\ln(n+1)}$, then the series (73.5) converges for all $x \in (0, \pi)$, although

$$\sum_{n=1}^{\infty} |c_n|^\alpha = \infty$$

for all real $\alpha > 0$.

20. If the series (73.5) converges absolutely at all the points of some set E with $mE > 0$, then the series (73.6) converges absolutely.

[This statement is proved in the same way as the Lusin–Denjoy theorem of § 61.]