Major theorems

Theorem 1 (Main Thm AC)**.** *Let* $\varepsilon > 0$ *and* $p \in (0,1)$*. There exists a constant* $c = c(p,\varepsilon) > 0$ *such that for* $\lambda \in \overline{Q} \cap (0,1)$*, if*

$$1 - \lambda < c \min\left\{\log M_\lambda, \frac{1}{(\log M_\lambda)^{1+\varepsilon}}\right\},$$

then $\mu_\lambda \ll \mathcal{L}$ *with* $d\mu_\lambda/d\mathcal{L} \in L \log L$*.*

Theorem 2 (CEB)**.** *There exists* $C = 1.2 \times 10^7$ *such that: Let* $\mu, \nu \in \mathcal{M}_c(\mathbb{R})$*. Let* $\alpha \in (0, 1/2)$ *and* $r > 0$*. Suppose for all* s *with* $|\log r - \log s| \leq 3 \log \alpha^{-1}$*,*

$$1 - H(\mu; s|2s) \leq \alpha \quad \text{and} \quad 1 - H(\nu; s|2s) \leq \alpha.$$

Then

$$1 - H(\mu * \nu; r|2r) \leq \frac{C}{(\log \alpha^{-1})^3}\alpha^2.$$

Theorem 3 (PHEP)**.** *Let* $\alpha \in (0, 1/2)$*. There exists* $c_0 = \frac{1}{625 \log \alpha^{-1}}$ *such that: Let* $\mu, \nu \in \mathcal{M}_c(\mathbb{R})$*. Let* $\sigma_2 < \sigma_1 < 0$ *and* $0 < \beta < 1/2$*. Suppose*

(a) $\mathcal{N}_1\{\sigma \in [\sigma_2, \sigma_1] : H(\mu; 2^\sigma | 2^{\sigma+1}) > 1 - \alpha\} < c_0 \beta (\sigma_1 - \sigma_2)$.

(b) $H(\nu; 2^{\sigma_2} | 2^{\sigma_1}) \geq \beta(\sigma_1 - \sigma_2)$.

Then, by setting $c = \frac{1}{1.35 \times 10^6} \frac{\alpha}{\log \alpha^{-1}}$*, we have*

$$H(\mu * \nu; 2^{\sigma_2}|2^{\sigma_1}) \geq H(\mu; 2^{\sigma_2}|2^{\sigma_1}) + c\frac{\beta}{\log \beta^{-1}}(\sigma_1 - \sigma_2) - 3.$$

Key ingredients to Theorem 1

Proposition 4 (CEB-kHE)**.** *Let* $\mu, \nu \in \mathcal{M}_c(\mathbb{R})$*. Let* $r > 0$ *and* k *with* $0 \leq k \leq 1 + \log^{(3)} r^{-1}$*. Suppose* A *large enough and* $r = r(A, C)$*, where* C *is in Theorem 2, small enough.*

If μ, ν *are* k*-HE at scale* r*, then* $\mu * \nu$ *is* $(k+1)$*-HE at scale* r*.*

Lemma 5 (PHEP-SingleScales)**.** *Let* $\alpha \in (0, 1/2)$ *and* $0 < p < 1$*. There exists* $c = (\alpha, p) > 0$ *such that: for* $\lambda \in \overline{Q} \cap (1/2, 1)$*, suppose the (scale)* $\tau = \tau(\lambda, p, \alpha)$ *sufficiently small and choose (auxilliary free measure scale)* $t \in (0,1)$ *with*

(\$) $\lambda > 1 - c\dfrac{\min\{1, \log M_\lambda\} \log t}{\log^{(2)}(M_\lambda + 2) \log \tau}$

(£) $t \geq \tau^{1/6}$.

Then there exists (counting) $K \geq c K_\lambda^{-1} \log \tau^{-1}$ *and (scales)* $\tau^{c/\log(M_\lambda+1)} > s_1 > \cdots > s_K > \tau$ *such that* $s_i > 2s_{i+1}$ *(disjointness) and*

$$H(\mu^{(t^2,t)}; s_i|2s_i) > 1 - \alpha \quad \text{for all } i.$$